UKMT

United Kingdom
Mathematics Trust

Mentoring Scheme

Supported by OxFORD ASSET MANAGEMENT

Hypatia

Sheet 1

Solutions and comments

This programme of the Mentoring Scheme is named after Hypatia of Alexandria (c. 370–415 CE).

See http://www-history.mcs.st-and.ac.uk/Biographies/Hypatia.html for more information.

Enquiries about the Mentoring Scheme should be sent to:

Mentoring Scheme, UK Mathematics Trust, School of Mathematics,
University of Leeds, Leeds LS2 9JT

☎ 0113 343 2339 mentoring@ukmt.org.uk www.ukmt.org.uk

Version 1.1, Oct 2019

Hypatia was the the daughter of the mathematician and philosopher Theon. She was born about 370 CE and was murdered in March 415 by a mob convinced that mathematics was a form of pagan magic. Although none of her writings survive, it is known that she lectured in mathematics in Alexandria and became head of the Platonist school there around 400 CE. It seems that she assisted her father in writing commentaries on works of mathematics and astronomy; this included the production of a new edition of Euclid's *Elements*, which became the basis for all later editions. Euclids's *Elements* included sections on number theory as well as on geometry. Translations into Latin appeared in Europe at the start of the 12th century, the first by Adelhard from Bath, while there were earlier translations into Arabic which influenced mathematics in the Middle East for many centuries.

Euclid's geometry continues to be taught through to current times, but the teaching of Euclid's *Elements* itself has disappeared from British schools in recent decades. Question 4 is in recognition of geometrical ideas which took root in Greek mathematics, formalised by Eudoxus in the 4th century BCE between the time of Pythagoras and the time of Euclid.

1. A prime number p is chosen so that $2011 + p$ is a power of 2.

What is the minimum possible value of $27p$?

Answer 999

Solution

The first power of 2 after 2011, namely 2048, gives $p = 37$ which is prime. The next few powers of 2 do not give prime p. The next such power is $2^{15} = 32\,768$ giving $p = 30\,757$. It so happens that 30 757 is also a Fibonnacci number.

2. Work out the day of the week for 14th September 1752, the first day of the new calendar in England when an Act of Parliament brought the calendar into alignment with the Gregorian calendar used in most of Western Europe by that time. You need to know that leap years occur every 4 years with the exception that every 100th year is *not* a leap year. There is even an exception to that! Every 400th year *is* a leap year.

Answer Thursday

Solution

The last method we should use is to count the days in total! In 2019, 14th September was on a Saturday. Every year you go forward, the day advances by 1 through the week, except in leap years when it advances 2 through the week. Note that there are $2019 - 1752 = 267$ years counting 1753, 1754, . . . , 2019. This includes 64 leap years: $267/4 = 66.75$, which suggests 66 leap years, but 1800 and 1900 were not leap years and 2000 was a leap year. Also, February in the leap year 1752 had already passed, so that does not need to be taken into account. Thus the day of the week advanced 331 places between 14th September in 1752 and 2019. 331 is 2 more than a multiple of 7.

> In 1752, the Julian calendar in England finally being brought into alignment with the Greogorian calendar meant that the 2nd September 1752 was followed by the 14th September 1752. Not only this, the year number used to change on the 25th March (called Lady Day, 9 months before Christmas Day) and not on the 1st January. Confused? So too, on occasions, have historians got muddled up about this.
>
> Isaac Newton was born on Christmas Day in 1642 but at this time the Gregorian Calendar was 10 days ahead of the Julian Calendar. By today's calendar, we should count him born on the 4th January 1643. Charles II lost his head earlier that year. Thus historians have to be careful interpreting dates in original documents.
>
> The Gregorian calendar ensures that the calendar stays in alignment with the seasons so that the spring equinox stays close to 21st March. The problem was that the Julian calendar had slightly too many leap years to make this work. You could look up the Gregorian calendar (for example on Wikipedia) for more information on this topic. There is also a very detailed book called "Calendrical Calculations" by Rynhold and Dershowitz which discusses conversions between many different calendrical systems from around the world, including these.

3. You are given that n is a positive integer with the property that when we add n and the sum of its digits, we obtain the number 313. What are the possible values of n?

ANSWER 296 and 305

SOLUTION

The greatest digital sum (that is, the sum of the digits) of a 3-digit number less than 313 comes from the number 299 which has digital sum 20. Hence we can not have a solution less than 293.

Now

$$29c + \text{digital sum of } 29c = 301 + 2c,$$

where c is the units digit of the number. Taking $c = 6$, we find that 296 plus its digital sum is 313, as required. There are no other solutions less than 300.

In the same way,

$$30c + \text{digital sum of } 30c = 303 + 2c,$$

so taking $c = 5$ shows that 305 plus its digital sum is also 313. There are no other solutions less than 310.

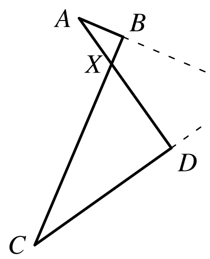
Finally, as 310 + digital sum of 310 = 314, there are no possibilities larger than 309.

4. In the sketch below, AD and BC intersect at X. $\angle ABX = 90°$. $\triangle ABX$ and $\triangle CDX$ are similar.

$AB = 4$, $AX = 5$ and $AD = 20$.

AB and CD extended intersect at Y, though Y is not shown on the diagram.

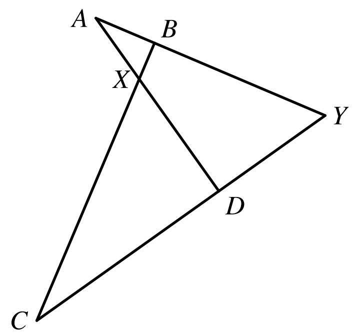
What is the area of $BXDY$?



ANSWER 144

SOLUTION

A completed diagram is needed.



First we note that we can use Pythagoras's Theorem to find BX:

$$AB^2 + BX^2 = AX^2$$
$$\text{so} \quad BX = 3.$$

(Or we could have noted that $\triangle ABX$ is a 3-4-5 triangle.)

$\triangle ADY$ is similar to $\triangle ABX$. This is because the angle at A is common to both triangles, both triangles contain a right-angle and therefore their third pair of angles are equal. Comparing ratios:

$$BX : BA = DY : DA$$
$$3 : 4 = DY : 20.$$
$$\text{Hence } DY = 15.$$
$$\text{Area of } \triangle ABX = \tfrac{1}{2} \times 3 \times 4 = 6.$$
$$\text{Area of } \triangle ADY = \tfrac{1}{2} \times 15 \times 20 = 150.$$

Complete by subtracting the areas.

We could also subtract the area of $\triangle CXD$ from the area of $\triangle CBY$.

> In a sense point C is a distractor. If $\triangle CDX$ had not been drawn, it would have made it more obvious that the relevant similar triangles were ABX and ADY.

5. A sequence of numbers is defined by the rule that each number is the sum of the one before it and the one after it. The 1st number in the sequence is -5 and the 26th is 2. Find the 12th number.

Answer -7

Solution

Suppose the first two numbers are x and y, where $x = -5$. The sequence is then:

$$x,\ y,\ y-x,\ -x,\ -y,\ x-y,\ x,\ y,\ y-x,\ \ldots$$

We see that sequence is *periodic* and recurs with period 6.

From this we can deduce that the 26th term is the same as the 2nd term, which is y. So $y = 2$.

The 12th term is then $x - y = -5 - 2 = -7$.

6. Let a and b be positive integers such that $b = \sqrt{a\sqrt{a\sqrt{a}}}$ and such that $a > 1$. Find the least possible value of $a + b$.

Answer 384

Solution

We see that by repeatedly squaring:

$$\begin{aligned} b &= \sqrt{a\sqrt{a\sqrt{a}}} \\ \text{so}\quad b^2 &= a\sqrt{a\sqrt{a}} \\ \text{so}\quad b^4 &= a^3\sqrt{a} \\ \text{so}\quad b^8 &= a^7 \end{aligned}$$

This will be satisfied by choosing $a = n^8$ and $b = n^7$ for some integer n.

We choose $n = 2$ to give the smallest possible integer values of a and b. So $a = 256$ and $b = 128$.

7. Show that

$$1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}\right) + \left(\tfrac{1}{9} + \tfrac{1}{10} + \cdots + \tfrac{1}{15} + \tfrac{1}{16}\right) > 3.$$

What happens to the sum of the fractions on the left as the number of terms increases?

Solution

Each bracketed expression exceeds $\frac{1}{2}$.

For example in $\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$, each fraction is at least $\frac{1}{8}$, and all but the last are greater than $\frac{1}{8}$.

Hence:

$$\begin{aligned}
&\ 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}\right) + \left(\tfrac{1}{9} + \tfrac{1}{10} + \cdots + \tfrac{1}{15} + \tfrac{1}{16}\right) \\
>&\ 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}\right) + \left(\tfrac{1}{16} + \tfrac{1}{16} + \cdots + \tfrac{1}{16} + \tfrac{1}{16}\right) \\
=&\ 3
\end{aligned}$$

As the number of terms increases (in the same way), the sum will grow without limit; it tends to infinity. This series is called the harmonic series.

8. How many pairs of distinct numbers can you choose from the set $\{1, 2, 3, 4, \ldots, 2018, 2019\}$ such that their sum is an even number?

Answer 1018 081

Solution

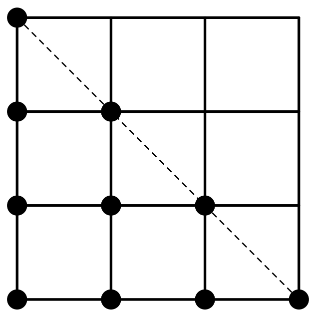
We must break this solution into two parts as we can make an even sum by picking either two odd or two even numbers. First consider the set $\{1, 3, 5, \ldots 2017, 2019\}$, which contains 1010 odd numbers (why?). We give two ways of counting the number of pairs of odd numbers.

Approach 1

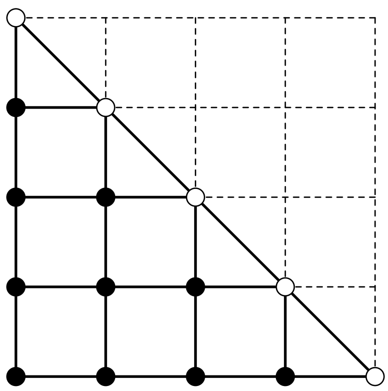
If we choose 1 as one of the numbers, we can choose any of the numbers in $\{3, 5, \ldots 2017, 2019\}$ to pair with 1. This gives 1009 possibilities. If our first choice is 3, we can choose any number from $\{5, 7, \ldots 2017, 2019\}$. Note that we cannot choose 1 with 3 because we have already counted that pair. This gives 1008 possibilities. Continuing this process, the total number of pairs is:

$$1009 + 1008 + 1007 + \cdots + 1$$

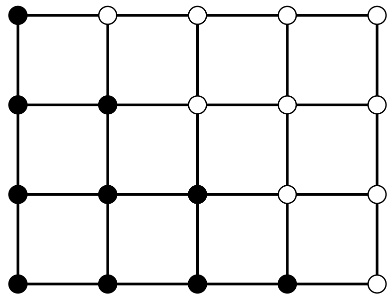
If you look at the figure below where the circles represent $4 + 3 + 2 + 1$, you can see we could work this out by taking a 4 by 4 square of circles (total 16), taking off the number on the diagonal (4), dividing the number of remaining circles (12) by 2 to get the number below the diagonal, then adding the diagonal again.



We could also take a 5 by 5 square of circles (total 25), take off the number on the diagonal (5) and divide the remaining number of circles (20) by 2.



As yet another way, we could lay two such grids of points next to each other, making a 4 by 5 rectangle of points:



which shows that $4 + 3 + 2 + 1 = \frac{1}{2} \times 4 \times 5$.

This leads quickly to the general formula

$$1 + 2 + 3 + \cdots + n = \tfrac{1}{2}n(n + 1).$$

We can now deduce that the number of pairs of odd numbers is $\frac{1}{2} \times 1009 \times 1010 = 509\,545$.

Approach 2

If we choose 1 as one of the numbers, we can choose any of the numbers in $\{3, 5, \ldots 2017, 2019\}$ to pair with 1. This gives 1009 possibilities. If our first choice is 3, we can choose any number from $\{1, 5, 7, \ldots 2017, 2019\}$, again giving 1009 possibilities. The same is true for whichever we pick as our first number, giving 1010×1009 pairs. However, we have counted every pair twice, for example choosing 7 and then 1083 is also counted when we choose 1083 first and then 7. So the number of different pairs is half of this total, $\frac{1}{2} \times 1010 \times 1009 = 509\,545$.

We now have to repeat this process for a pair of even numbers. There are 1009 even numbers in the set $\{2, 4, 6, \ldots 2016, 2018\}$, so there are $\frac{1}{2} \times 1008 \times 1009 = 508\,536$ pairs of even numbers, using the same reasoning as before.

Thus the total is $509\,545 + 508\,536$.

Another way to find the total is to look at the original calculations:

$$\tfrac{1}{2} \times 1009 \times 1010 + \tfrac{1}{2} \times 1008 \times 1009 = \tfrac{1}{2} \times 1009 \times (1008 + 1010) = 1009^2,$$

so the answer is actually a square number.

United Kingdom
Mathematics Trust

Mentoring Scheme

Supported by OxFORD ASSET MANAGEMENT

Hypatia

Sheet 2

Solutions and comments

This programme of the Mentoring Scheme is named after Hypatia of Alexandria (c. 370–415 CE).

See http://www-history.mcs.st-and.ac.uk/Biographies/Hypatia.html for more information.

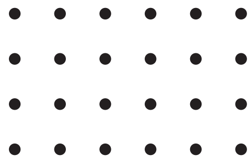
Enquiries about the Mentoring Scheme should be sent to:

Mentoring Scheme, UK Mathematics Trust, School of Mathematics,
University of Leeds, Leeds LS2 9JT

☎ 0113 343 2339 mentoring@ukmt.org.uk www.ukmt.org.uk

Version 1.2, Nov 2020

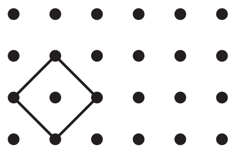
1. 24 points are arranged in a regular rectangular grid of 4 rows and 6 columns. How many rectangles or squares can be formed by joining four of the points so that the edges are not parallel to the rows or columns of the grid?



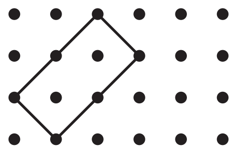
Answer 20

Solution

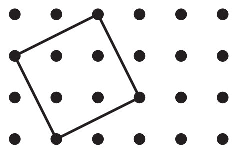
There are three possible shapes (up to congruence) as illustrated in the three diagrams below.



8 positions



2 orientations $\times$ 3 positions



2 orientations $\times$ 3 positions

If one of the sides has gradient less than $\frac{1}{2}$, the perpendicular sides must have absolute gradient at least 3, which is impossible on this grid.

2. Each of the four triangular faces of a regular tetrahedron is to be painted either red or green. How many different (distinguishable) ways are there of painting the tetrahedron? We allow the tetrahedron to be turned and viewed from any direction so, for example, painting the front face red and the rest green is considered the same as painting the bottom face red and the rest green.

Answer 5

Solution

The only case that might cause pause for thought is when there are 2 red faces and 2 green faces. But the red faces must have an edge in common, as must the green faces, so this still accounts for just one case up to rotation. There are thus 5 possibilities, with 0, 1, 2, 3 and 4 red faces respectively.

3. The first of the seven Millennium Problems in mathematics to be solved was the long-standing Poincaré Conjecture. Grigori Perelman succeeded in finding a proof in 2003.

Find the last three digits of 7^{2003}. (It can be done without a calculator.)

Answer 343

Solution

If you keep multiplying by 7 and ignoring anything which carries over into the thousands position, you will get back to the digits 001 after 20 multiplications. Thus 7^{2000} will end in 001 and 7^{2003} will have the same final three digits as 7^3.

You might also notice patterns in the last three digits: the unit digit cycles 7, 9, 3, 1, while the tens digit is 0 after every fourth power (so 7^4 ends 401, 7^8 ends 801, 7^{12} ends 201 and 7^{16} ends 601); this also shows that the hundreds digit increases by 4 every fourth power (wrapping around from 8 to 2 and from 6 to 0). These patterns can also help to make the calculations more efficient.

4. p, q and r are prime numbers with $p < q < r$ and

$$\frac{p}{q} + \frac{q}{r} = \frac{s}{qr}.$$

Prove that if s is prime, then $p = 2$.

Solution

We can multiply both sides of the equation by qr to get

$$s = pr + q^2. \tag{1}$$

The only even prime is 2. If $p > 2$, then p, q and r are all odd, so pr and q^2 are both odd, and $s = pr + q^2$ is even, thus s cannot be prime (as $pr + q^2 > p > 2$).

Therefore we must have $p = 2$.

If $p = 2$, there are values of q and r which make s prime, for example if $q = 3$ and $r = 5$, then $s = 19$.

5. A 10 digit number $abcdefghij_{10}$ includes all the digits 0 to 9. How many such numbers are there such that $a + j = b + i = c + h = d + g = e + f = 9$?

[Note that by $abcdefghij_{10}$, we mean the decimal (base 10) number with ones digit j, tens digit i, hundreds digit h and so on.]

Answer 3456

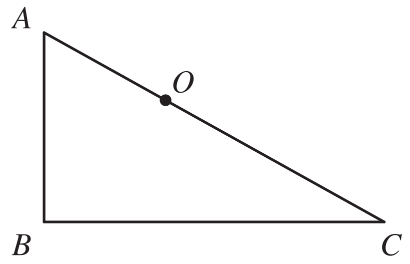
Solution

There are 9 possibilities for a and j since $a \neq 0$. This leaves 8 possibilities for b and i for each way that a and j are chosen. This reasoning continues with 6 possibilities for c and h, 4 possibilities for d and g and just 2 possibilities for e and f. Multiply these numbers together.

Alternatively, there are 5 pairs of digits which sum to 9 (0 and 9, 1 and 8, and so on) so there are 5! ways of assigning these to a and j, b and i, and so on. In each assignment they can be done in 2 orders so there are $5! \times 2^5$ ways. But we must exclude the case where $a = 0$, $j = 9$ for which there are $4! \times 2^4$ possibilities for the remaining digits (by similar reasoning). We deduce that there are

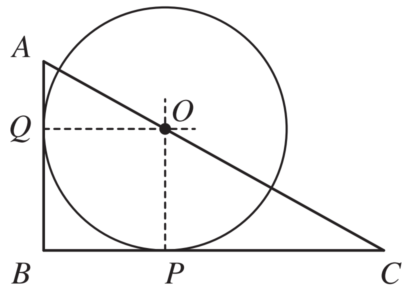
$$5! \times 2^5 - 4! \times 2^4 = 3840 - 384 = 3456 \text{ numbers.}$$

6. In $\triangle ABC$, $AB = 3$, $BC = 4$ and $\angle ABC = 90°$, as shown in the sketch. A circle has its centre O on AC and touches AB and BC. What is the radius of the circle?



ANSWER $\frac{12}{7}$

SOLUTION



Construct OP and OQ as suggested in the hint. These are radii of the circle and meet AB and BC at 90°. They are also equal in length, so $BPOQ$ is a square. We note that OQ is parallel to BC so $\angle AOQ = \angle ACB$. We have sufficient information to state that $\triangle AQO$ is similar to $\triangle ABC$.

Let r be the radius of the circle. Then $AQ = 3 - r$, $QO = r$, $AB = 3$ and $BC = 4$. Hence by ratios of corresponding sides of similar triangles

$$\begin{aligned} \frac{3-r}{r} &= \frac{3}{4} \\ \text{so} \quad 12 - 4r &= 3r \\ \text{so} \quad 12 &= 7r \end{aligned}$$

Note that there are three similar triangles in the figure but we only need to use two of them. You might note that the result can be checked by doing a similar calculation for $\triangle OPC$.

7. On her penultimate maths test of the school year, Barbara scored 98 and her average (mean) score so far then increased by 1. On her final maths test she scored 70, causing her most recent average score to decrease by 2. How many maths tests did she take during the school year?

Answer 10

Solution

Let n be the number of tests and a her average score before she took the last two tests. Then we can set up the following equations:

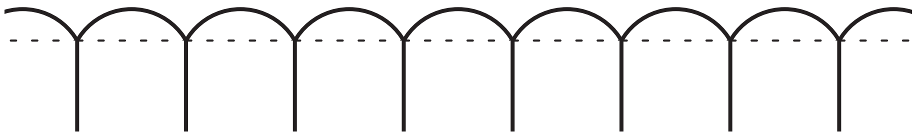
$$(n-1)(a+1) = (n-2)a + 98 \qquad \text{for the second to last test}$$
$$n(a-1) = (n-2)a + 98 + 70 \qquad \text{for the last test}$$

These simplify to:

$$2n + a = 99$$
$$2a - n = 168.$$

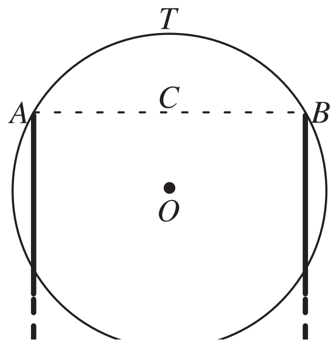
Adding these equations gives $3a = 267$, so $a = 89$ and $n = 10$.

8. A blacksmith is building a fence consisting of uprights 18 cm apart. Instead of welding a bar across the top of two uprights (like the dotted line), he makes individual arcs of circles like the ones shown in the diagram. (These are *not* semi-circles.) He continues this pattern along the fence. The highest point of the arc is $3\sqrt{3}$ cm above the dotted line. All the pieces of metal lie in the same vertical plane. Treat each piece of metal as being very thin. How long is the piece of metal used to make one of the circular arcs?



ANSWER $4\pi\sqrt{3}$

SOLUTION



Let T be the top point of the arc, A and B the ends of the dotted line, C the centre of the dotted line and O the centre of the arc.

We are given $CT = 3\sqrt{3}$ and $BC = 9$.

We can use Pythagoras's Theorem to calculate $BT^2 = (3\sqrt{3})^2 + 9^2 = 27 + 81 = 108$.

This gives $BT = \sqrt{108} = \sqrt{36 \times 3} = 6 \times \sqrt{3} = 2 \times CT$.

Therefore $\triangle BTC$ is half an equilateral triangle. Because OB and OT are radii of the circle, they are equal, and so $\triangle BOT$ must be an equilateral triangle. Therefore $\angle BOT = 60°$. It follows that the arc of the circle ATB is exactly one-third of the complete circle.

The radius of the circle is $6\sqrt{3}$ so the circumference of the circle is $2\pi \times 6\sqrt{3}$ and the length of the arc ATB is one-third of this.

An alternative is to use algebra. Let $OB = OT = r$. Apply Pythagoras's Theorem on $\triangle OBC$:

$$\begin{aligned} & r^2 = (r - 3\sqrt{3})^2 + 9^2 \\ \text{so} \quad & r^2 = r^2 - 6r\sqrt{3} + 27 + 81 \\ \text{so} \quad & 6r\sqrt{3} = 108 \\ \text{so} \quad & r = \frac{18}{\sqrt{3}} = \frac{6 \times 3}{\sqrt{3}} = 6\sqrt{3} \end{aligned}$$

To find the angle $\angle BOT$, we could then use trigonometry on the triangle $\triangle BOC$ or note once again that it is half of an equilateral triangle.

United Kingdom
Mathematics Trust

Mentoring Scheme

Supported by OxFORD ASSET MANAGEMENT

Hypatia

Sheet 3

Solutions and comments

This programme of the Mentoring Scheme is named after Hypatia of Alexandria (c. 370–415 CE).

See http://www-history.mcs.st-and.ac.uk/Biographies/Hypatia.html for more information.

Enquiries about the Mentoring Scheme should be sent to:

Mentoring Scheme, UK Mathematics Trust, School of Mathematics,
University of Leeds, Leeds LS2 9JT

☎ 0113 343 2339   mentoring@ukmt.org.uk   www.ukmt.org.uk

Version 1.1, Dec 2019

1. How many integer values can

$$\frac{100}{2n-1}$$

take if n is a positive integer?

ANSWER 3

SOLUTION

$2n - 1$ is an odd number if n is a positive integer. The only odd factors of 100 are 1, 5 and 25 (for which $n = 1$, 3 and 13 respectively).

2. In a magic square, the sum of the three numbers in any row, column or diagonal is the same. In the magic square shown, determine the value of $B + E$.

5	A	B
C	8	D
E	F	G

ANSWER 16

SOLUTION

Let S be the sum of a row, column or diagonal. By adding the two diagonals, the second row and the second column together we obtain:

$$A + B + C + D + E + F + G + 37 = 4S.$$

By adding all the rows together:

$$A + B + C + D + E + F + G + 13 = 3S.$$

Hence $S = 24$. We can now use a diagonal to deduce the value of $B + E$.

If the 5 and 8 are replaced by two variables, this argument shows that for a 3×3 magic square, the sum of each row, column and diagonal is equal to three times the centre square.

3. A snail climbs up a cylindical column 8 m high whose base circumference is 3 m. Starting from the base it goes round the column twice reaching the top vertically above the point at which it started. What is the shortest length of path the snail could have taken?

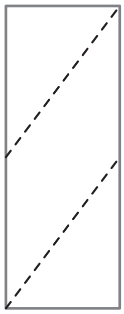
Answer 10 m

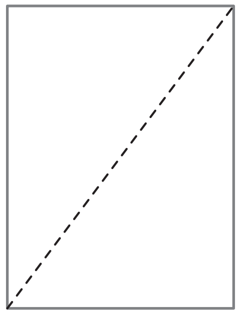
Solution

There are two ways we could wrap the paper round the cylinder then imagine the paper unwrapped.

If we wrap the paper round once, then unwrapped it will form a rectangle 8 m by 3 m as shown in the left hand diagram (overleaf). The shortest journey the snail can take is represented by the broken diagonal line in the diagram. If l is the length of half the track, then $l^2 = 3^2 + 4^2$.

If we wrap the paper round twice, then unwrapped it will form a rectangle 8 m by 6 m as shown in the right hand diagram. The shortest journey the snail can take is represented by the diagonal line in the diagram. If L is the length of the full track, then $L^2 = 6^2 + 8^2$.





4. You are given a set of thin stiff rods which have integer lengths from 1 to 10. Consider all possible choices of three rods you can make whose total length is 12. (There are enough of each length to use any particular length as many times as you wish.) For what fraction of these choices can the rods be laid out to form a triangle in such a way that each end of a rod touches the end of another rod?

In this question, order does not matter: for example, choosing 2, 2, 8 is considered the same as choosing 8, 2, 2.

ANSWER $\frac{1}{4}$

SOLUTION

We start by listing the *partitions* of 12 into 3 integers, that is, 3 integers which sum to 12, listing the three integers in increasing size, and then listing the partitions in "alphabetical" order:

1	1	10
1	2	9
1	3	8
1	4	7
1	5	6
2	2	8
2	3	7
2	4	6
2	5	5
3	3	6
3	4	5
4	4	4

There are 12 of them but only $(2, 5, 5)$, $(3, 4, 5)$ and $(4, 4, 4)$ can be used to form a triangle.

This could lead to discussion of the triangle inequality. If the three sides of a triangle are a, b, c, then it follows that $a + b > c$, $b + c > a$ and $c + a > b$. The converse is also true, namely that if $a + b > c$, $b + c > a$ and $c + a > b$, then these lengths can form a triangle.

The sets $(1, 5, 6)$, $(2, 4, 6)$ and $(3, 3, 6)$ form *degenerate* triangles: all three vertices lie on a straight line.

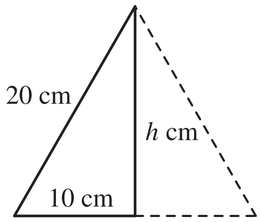
The formal name for the order in which we have listed these partitions is *lexicographic order*.

5. An equilateral triangle whose edges measure 20 cm is dissected and rearranged to form a square without overlap or gaps. How long is the side of the square?

To see how this can be done, you could look at http://mathworld.wolfram.com/Dissection.html

Answer $\sqrt{100\sqrt{3}}$ cm

Solution



Applying Pythagoras's Theorem to the triangle shown, we have

$$10^2 + h^2 = 20^2$$
$$\text{so} \qquad h^2 = 300$$

Therefore the area of the triangle is

$$\tfrac{1}{2} \times \text{base} \times \text{height} = \tfrac{1}{2} \times 20 \times \sqrt{300} = 10 \times 10\sqrt{3} = 100\sqrt{3}$$

and the side of the square is the square root of this.

> Note that $\sqrt{\sqrt{3}}$ can also be read as the fourth root of 3 written as $\sqrt[4]{3}$ or $3^{\frac{1}{4}}$.
>
> Other ways of writing this answer include $10\sqrt[4]{3}$, $10 \times 3^{\frac{1}{4}}$, $10\sqrt{\sqrt{3}}$ and $\sqrt{\sqrt{30000}}$.

6. How many integer solutions (x, y) are there to the equation $(x + y)^2 + 12 = x^2 + y^2$?

If the equation were still $(x + y)^2 + 12 = x^2 + y^2$ and you were allowed to use any *real* numbers for (x, y), how would this change your answer (if at all)?

Answer 8

Solution

We can write

$$\begin{aligned} & (x+y)^2 + 12 = x^2 + y^2 \\ \iff \quad & x^2 + 2xy + y^2 + 12 = x^2 + y^2 \\ \iff \quad & 2xy + 12 = 0 \\ \iff \quad & xy = -6 \end{aligned}$$

It is important to note that each line above is equivalent to its neighbour above or below, and this is indicated by the use of the $\iff$ symbol between the lines. We therefore do not lose or gain any solutions in the process of manipulating the algebra.

Now we must list the (x, y) pairs which fit this equation. They are:

$$(-6, 1), (-3, 2), (-2, 3), (-1, 6), (1, -6), (2, -3), (3, -2), (6, -1)$$

For the second part, the equation is still equivalent to

$$xy = -6$$

(as we did not assume that x and y were integers in our original manipulations), so there are an infinite set of real solutions. The graph of $y = -\frac{6}{x}$ passes through all the integer points you have found, and is a curve with two branches approaching the axes as you move away from the origin. You could explore this graph on paper or by using software such as GeoGebra to draw it.

7. A five-digit number N has the following property. The number obtained by writing the digit 1 after the final digit of N (to make a six digit number) is three times the number obtained by writing the digit 1 before the first digit of N. What is N?

Answer 42 857

Solution

A quick way to do this is to use algebra. We have

$$\begin{aligned} 10N + 1 &= 3(100\,000 + N) \\ \text{so} \quad 10N + 1 &= 300\,000 + 3N \\ \text{so} \quad 7N &= 299\,999 \\ \text{hence} \quad N &= 42857. \end{aligned}$$

An alternative is to set out the problem as a traditional multiplication, showing $N \times 3$:

$$\begin{array}{rcccccc} & 1 & a & b & c & d & e \\ \times & & & & & & 3 \\ \hline & a & b & c & d & e & 1 \end{array}$$

We can now see that we require $3 \times e = 10t + 1$ where t is the tens unit from the muliplication. The only possible value for e is 7.

Now we require the multiplication $3 \times d$ plus a carry of 2 to yield a units digit 7. Hence $d = 5$.

Next we require the multiplication $3 \times c$ plus a carry of 1 to yield a units digit 5. Hence $c = 8$.

We now have the multiplication $3 \times b$ plus a carry of 2 gives a units digit 8. Hence $b = 2$.

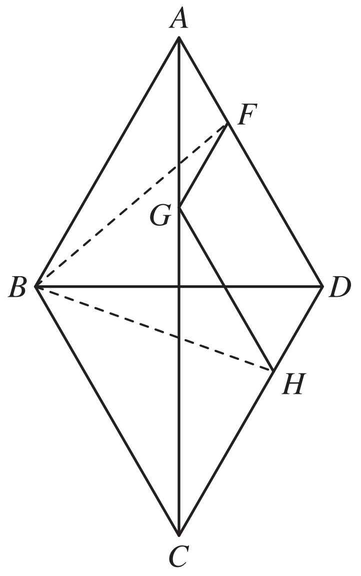
Finally, the multiplication $3 \times a$ (with no carry) has a units digit 2. Hence $a = 4$.

We also check that 3×1 plus the carry of 1 does give 4.

Try working out the values of $\frac{1}{7}$ and $\frac{3}{7}$ as recurring decimals. What do you notice?

8. A rhombus $ABCD$ has $\angle BCD = 60°$. An arbitrary point G is chosen on the diagonal AC, then points F and H are marked on sides AD and DC respectively so that $DFGH$ is a parallelogram. Show that $CH = DF$. Also prove that $\triangle FBH$ is equilateral.

Would either of these results still be true if $\angle BCD$ were not equal to 60°?



Solution

$\angle ACD = \angle CAD = \angle ACB = \angle CAB$ since the diagonal AC bisects the angles at the vertices A and C of the rhombus. GH is parallel to AD so $\angle HGC = \angle DAC$. Hence $\angle HGC = \angle HCG$ and so $\triangle HGC$ is isosceles, giving $CH = GH$; since $DF = HG$ because $DFGH$ is a parallelogram, it follows that $CH = DF$.

(Similarly we can show that $AF = FG = DH$ but this is not needed to continue the proof.)

Next consider $\triangle BCH$ and $\triangle BDF$. $ABCD$ is a rhombus with $\angle ABC = 120°$, so $\angle ABD = 60°$. It follows that $\triangle BCD$ and $\triangle ABD$ are equilateral. We thus have $BC = BD$, $\angle BCH = \angle BDF = 60°$, and we have already proved $CH = DF$; it follows by the SAS rule that $\triangle BCH$ is congruent to $\triangle BDF$. Hence $BF = BH$.

For the last step, we have $\angle FBH = \angle FBD + \angle DBH = \angle FBD + 60° - \angle HBC = 60°$ because we have proved that $\angle FBD = \angle HBC$. Together with $BF = BH$, this is sufficient to show that $\triangle BFH$ is equilateral.

Alternatively, we could have shown that $\triangle ABF$ is congruent to $\triangle DBH$ and proceeded with a very similar proof to that above.

Finally, if $\angle BCD$ is not equal to 60°, the first part is still true, as we did not need to make any assumptions about angles to show that $CH = DF$. We can also deduce, in the same way, that $\angle FBH = \frac{1}{2} \times \angle ABC$, but this will not equal 60°, so $\triangle BFH$ will be isosceles but not equilateral.

United Kingdom
Mathematics Trust

Mentoring Scheme

Supported by OxFORD ASSET MANAGEMENT

Hypatia

Sheet 4

Solutions and comments

This programme of the Mentoring Scheme is named after Hypatia of Alexandria (c. 370–415 CE).

See http://www-history.mcs.st-and.ac.uk/Biographies/Hypatia.html for more information.

Enquiries about the Mentoring Scheme should be sent to:

*Mentoring Scheme, UK Mathematics Trust, School of Mathematics,
University of Leeds, Leeds LS2 9JT*

☎ 0113 343 2339 mentoring@ukmt.org.uk www.ukmt.org.uk

Version 1.2, Jan 2021

1. The integers m and n satisfy the equation $7m - 5n = 13$. Find one possible pair of values for m and n. Show that there are infinitely many pairs (m, n) of solutions to this equation, and find all such pairs.

Answer $m = 4$, $n = 3$ is one example; $(m, n) = (4 + 5k, 3 + 7k)$

Solution

One solution can be found by trial and error; the solution $m = 4$, $n = 3$ is the easiest to find, as it is the smallest positive integer solution.

We can look for other integer solutions by drawing the graph of $7y - 5x = 13$ and looking for integer points that lie on this line. That might give some ideas as to what is going on, which you can then try to justify. Here is an algebraic solution.

If (m, n) is another solution, then we have

$$7m - 5n = 13$$
$$7 \times 4 - 5 \times 3 = 13.$$

Subtracting these equations gives

$$7(m - 4) - 5(n - 3) = 0$$

or

$$7(m - 4) = 5(n - 3).$$

So $m - 4$ must be a multiple of 5, as 7 and 5 have no common factors (other than 1). Let us therefore write $m - 4 = 5k$, where k is an integer. The equation then becomes

$$5(n - 3) = 7 \times 5k = 35k$$

so that $n - 3 = 7k$, which is also an integer.

If we take $m = 4 + 5k$ and $n = 3 + 7k$ for any integer k, a quick piece of algebra shows that $7m - 5n = 13$, as we require. Hence these are all of the integer solutions.

> An equation such as this where we ask about all of the integer solutions is sometimes known as a *Diophantine equation*.

2. A teacher invited a group of children from two schools, Squareton Primary (SP) and Circletown Primary (CP), to sit down at a round table. There were three times as many SP pupils as CP pupils.

The teacher noted that there were twice as many pairs of children sitting next to each other who came from the same school as pairs who came from different schools. What is the smallest possible number of children that could be sitting at the table? Show how they could be be sitting.

Note that if there are n children, there are n pairs of children sitting next to each other.

Answer 12

Solution

From the second sentence of the question, we see that the total number of children must be a multiple of 4. From the third sentence, the number of children must be a multiple of 3. Combining these, the total number of children must be a multiple of 12. If we can find a suitable arrangement of 12 children satisfying these conditions, then this will be the smallest possible number of children.

In this case, we would have 9 SP children and 3 CP children. We would need to have 4 pairs where an CP child is sitting next to a SP child, and 8 pairs where a child is sitting next to another from their own school. If no two CP children are sitting next to each other, there would be 6 pairs of an CP child next to a SP child. If all three CP children were sitting together, there would only be 2 such pairs. But if 2 CP children are sitting next to each other while the other CP child is separate from them, we obtain exactly this arrangement.

3. (a) In our normal decimal system, the number written as 234 stands for $2\times10^2+3\times10+4$. We can also write this as 234_{10} to make the decimal system explicit.

In base 8 (octal arithmetic), we only use the digits 0 to 7, and the number written as 157_8 stands for $1 \times 8^2 + 5 \times 8^1 + 7$; the 7 is still in the 1s column, but now the 5 is in the 8s column and the 1 is in the 64s column. This number equals 111 in the decimal system.

In which base or bases b is it true that $15_b \times 15_b = 321_b$?

(b) What can you say about the base b if $11_b \times 11_b = 121_b$?

If you would like to investigate the idea of bases further, do visit the wonderful Exploding Dots website: explodingdots.org

Answer (a) $b = 6$; (b) $b > 2$

Solution

(a) To have a ones (units) digit of 1 in the answer requires $5 \times 5 = 25$ divided by the base to leave a remainder of 1. Another way of saying this is that 24 must be a multiple of the base. The base must also exceed 5, as there is a ‘5’ digit in the calculation. The possibilities for the base are therefore 6, 8, 12 and 24. It is straightforward to check by calculation that 6 works, but 8, 12 and 24 do not.

(b) This calculation works in almost any base, but because of the presence of the digit 2 in the answer, the base must exceed 2.

> When the base exceeds 10, we need extra symbols to represent digits. An important base for computing is base 16, known as hexadecimal. The extra digits for 10, 11, ..., 15 are denotated as A, B, C, D, E and F or a, b, c, d, e and f. For example, in HTML or CSS, colours are specified in hexadecimal: '#a080c0' means $a0_{16} = 160$ units of red (out of 255), $80_{16} = 128$ units of green and $c0_{16} = 192$ units of blue, which is a sort of lilac colour.

4. Alice and Bob are riding on a merry-go-round at a fairground. Alice sits on a horse on the inner ring which completes a revolution in 20 seconds. Bob sits on a horse on the outer ring which moves at a different speed and takes 28 seconds to complete a revolution. At a certain moment they are next to each other. After how many seconds are they on opposite sides of the centre of the merry-go-round?

Answer 35 seconds

Solution

After 20 seconds, Bob has moved $\frac{20}{28}$ of a revolution, so he is now $\frac{8}{28} = 27$ of a revolution behind Alice.

Hence the number of seconds after which Bob will be $\frac{1}{2}$ a revolution behind Alice is

$$\begin{aligned}\left(\tfrac{1}{2} \div \tfrac{2}{7}\right) \times 20 &= \tfrac{1}{2} \times \tfrac{7}{2} \times 20\\ &= 35.\end{aligned}$$

Alternatively, we can reason the other way round: after 28 seconds, Alice has moved $\frac{28}{20}$ of a revolution, so she is now $\frac{8}{20} = \frac{2}{5}$ of a revolution ahead of Bob.

Hence the number of seconds after which Alice will be $\frac{1}{2}$ a revolution ahead of Bob is

$$\begin{aligned}\left(\tfrac{1}{2} \div \tfrac{2}{5}\right) \times 28 &= \tfrac{1}{2} \times \tfrac{5}{2} \times 28\\ &= 35.\end{aligned}$$

5. Solve for x:

$$\frac{x+1}{1} + \frac{x+2}{2} + \frac{x+3}{3} + \cdots + \frac{x+100}{100} = 100$$

Answer $x = 0$

Solution

The equation can be rewritten as:

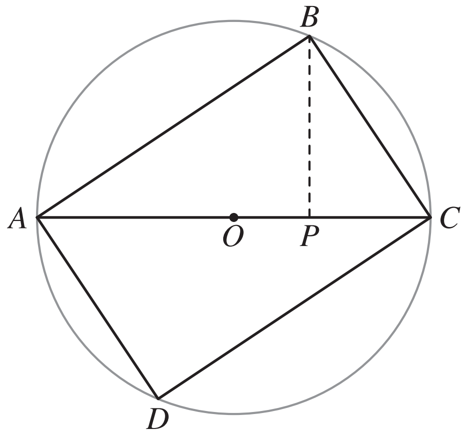
$$(x+1) + (\tfrac{1}{2}x + 1) + (\tfrac{1}{3}x + 1) + \cdots + (\tfrac{1}{99}x + 1) + (\tfrac{1}{100}x + 1) = 100.$$

Since there are 100 brackets here, we can subtract 100 from both sides to get

$$x + \tfrac{1}{2}x + \tfrac{1}{3}x + \cdots + \tfrac{1}{99}x + \tfrac{1}{100}x = 0$$
$$\text{so} \qquad (1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{99} + \tfrac{1}{100})x = 0$$

It does not matter what the fractions inside the bracket add up to; it is just some positive number. Therefore $x = 0$.

6. $ABCD$ is a rectangle. O is the centre of the rectangle and also of the circle through A, B, C and D. P lies on AC so that $\angle BPC = 90°$. Prove that $\triangle CBP$ is similar to $\triangle ACD$.



SOLUTION

Let $\angle CAD = x$. Because $\triangle ACD$ is right-angled at D, it follows from the angle sum of a triangle that $\angle ACD = 90° - x$. Because $\angle BCD = 90°$ too, we find that $\angle ACB = x$.

An alternative approach to showing this is to note that AD is parallel to BC because $ABCD$ is a rectangle, so $\angle ACB = x$ by alternate angles.

It is given $\angle BPC = 90° = \angle ADC$, so we also have $\angle ACD = \angle CBP$ by the angle sum of a triangle. Hence $\triangle CBP$ is similar to $\triangle ACD$.

Note that we did not need to make use of the circle here.

7. All of the sides in the non-regular heptagon $ABCDEFG$ have length 2.

Furthermore, $\angle DEF = 120°$, $\angle BCD = \angle FGA = 90°$ and $\angle GAB = \angle ABC = \angle CDE = \angle EFG$.

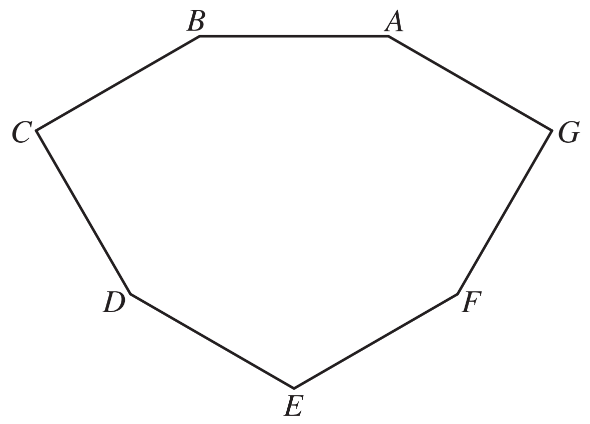
What is the area of the heptagon?

ANSWER $8 + 3\sqrt{3}$

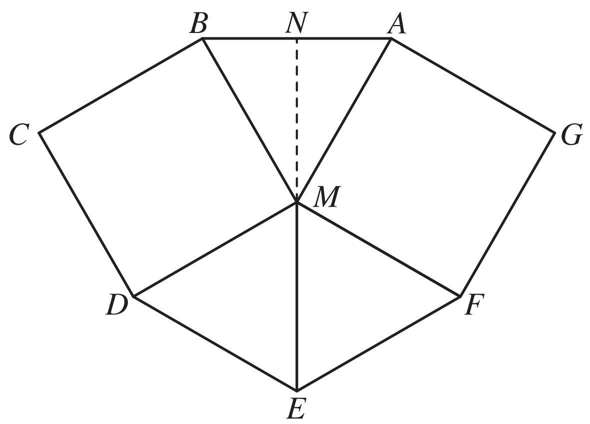
SOLUTION

First we draw a rough diagram. We start by working out the size of the four equal angles. The sum of the internal angles of an heptagon is $(7 - 2) \times 180° = 900°$. Thus the four equal angles must add up to $900° - 120° - 2 \times 90° = 600°$, so each is $150°$.

The heptagon therefore looks like this:



With some effort, it can now be seen that the heptagon can be broken into two squares and three equilateral triangles which have a common point M.



Each square has area 4 (as the side length is 2).

Consider $\triangle ABM$, and let N be the midpoint of AB. $\triangle ANM$ is right-angled at N. Hence we can use Pythagoras's theorem to deduce that

$$AN^2 + MN^2 = AM^2.$$

As $AN = \frac{1}{2}AM = 1$, so $MN^2 = 3$. Now $MN = \sqrt{3}$ is the height of the $\triangle ABM$ if we take AB as base. Hence $\triangle ABM$ has area $\frac{1}{2} \times 2 \times \sqrt{3} = \sqrt{3}$.

Thus each equilateral triangle in the diagram has area $\sqrt{3}$, and the total area is $8 + 3\sqrt{3}$.

> The method of finding the area of an equilateral triangle is worth learning.

8. Een, vier and negen are Dutch for 1, 4, 9. The numbers EEN_{10}, $VIER_{10}$ and $NEGEN_{10}$ are also squares, where the letters E, N, V, I, R and G represent six different decimal digits.

Find $\sqrt{EEN_{10}}$, $\sqrt{VIER_{10}}$ and $\sqrt{NEGEN_{10}}$.

As in question 3, EEN_{10} means the number with hundreds digit E, tens digit E and ones digit N.

Answer 21, 57 and 121

Solution

We will drop the suffix $_{10}$ in the calculations which follow.

All squares (in decimal notation) end in 0, 1, 4, 5, 6 or 9. The value of $\sqrt{EEN}$ must be less than 32, as $32^2 = 1024$, so we only have 22 possibilities to check; we fairly quickly find that EEN could only be $15^2 = 225$ or $21^2 = 441$.

If we take $E = 2$ and $N = 5$, then $NEGEN = 52G25$. As $228^2 = 51984$, $229^2 = 52441$ and $230^2 = 52900$, we see that no value of G makes this square. Hence $E = 4$ and $N = 1$, giving $EEN = 21^2$. Now $NEGEN = 14G41$, and $G = 6$ makes this a square: $NEGEN = 121^2$.

Now in $VIER$, R cannot be 0, because squares of multiples of 10 end in two zeros. We are left with R being 5 or 9. Squares of numbers ending in 5 end in 25, but $E = 4$, so $R \neq 5$. Hence $R = 9$.

We therefore have $VIER = VI49$, so $VIER = (X3)^2$ or $VIER = (X7)^2$. Here, X is a digit to be determined so that V and I are two distinct digits from the remaining possibilities 0, 2, 3, 5, 7 or 8. X must be at least 3 for $(X3)^2$ or $(X7)^2$ to be a four digit number.

A direct approach would be to calculate all the possibilities. This will give the required value of $VIER$.

We can be more sophisticated by noting that if

$$VI49 = (10X + 3)^2 = 100X^2 + 60X + 9,$$

we would need $X = 4$ or $X = 9$ to make the tens digit work. Similarly, if

$$VI49 = (10X + 7)^2 = 100X^2 + 140X + 49$$

we would need $X = 5$.

Now $43^2 = 1849$, $93^2 = 8649$ and $57^2 = 3249$, and the only one of these which does not use an already-used digit is 3249, so $VIER = 57^2$.

United Kingdom
Mathematics Trust

Mentoring Scheme

Supported by Oxford Asset Management

Hypatia

Sheet 5

Solutions and comments

This programme of the Mentoring Scheme is named after Hypatia of Alexandria (c. 370–415 CE).

See http://www-history.mcs.st-and.ac.uk/Biographies/Hypatia.html for more information.

Enquiries about the Mentoring Scheme should be sent to:

*Mentoring Scheme, UK Mathematics Trust, School of Mathematics,
University of Leeds, Leeds LS2 9JT*

☎ 0113 343 2339 mentoring@ukmt.org.uk www.ukmt.org.uk

Version 1.1, Feb 2020

1. Two candles have the same length. The thicker one lasts 5 hours, the other 3 hours, and they are lit at the same time. After how many minutes will the thicker candle be 3 times longer than the thinner one?

Answer 150 minutes

Solution

We can work in hours or minutes. Here is a solution working in hours (which uses smaller numbers than when working in minutes).

We can assume that each candle is 15 units long, so that the thicker burns at 3 units per hour and the thinner at 5 units per hour. Let t be the number of hour for which the candles have been burning. Then the thicker candle has lost $3t$ units and the thinner has lost $5t$ units. We must therefore solve the equation

$$15 - 3t = 3(15 - 5t).$$

Expanding the brackets gives $15 - 3t = 45 - 15t$, so $12t = 30$. Thus $t = 2.5$, and 2.5 hours is 150 minutes.

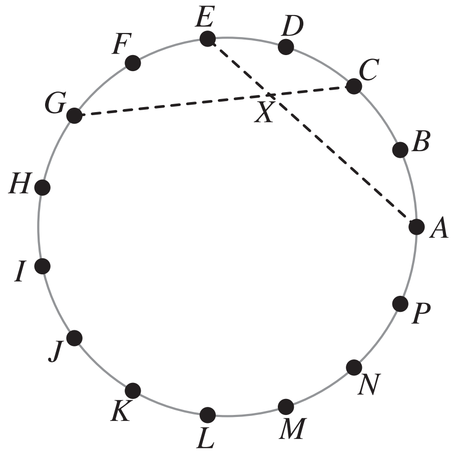
If instead we worked in minutes, the thicker candle lasts for 300 minutes and the thinner for 180 minutes. We can then take the length of the candle to be 900 units (900 being the least common multiple of 300 and 180), so that the thicker candle burns at 3 units per minute and the thinner at 5 units per minute. As before, this results in the equation

$$900 - 3t = 3(900 - 5t)$$

which gives $12t = 1800$ so $t = 150$.

> Note that we do not lose any generality in our solution by choosing a specific length for each candle; as the unit of length is not specified, it can be whatever is appropriate to make the length equal to 15 units. If we instead chose a letter to represent the length, this letter would cancel out of the equation.

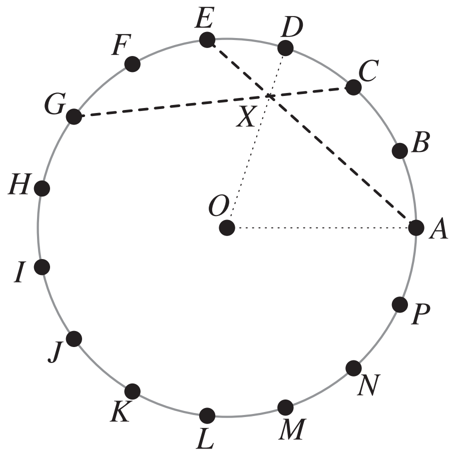
2. The vertices of a regular 15-sided polygon $ABCD \dots MNP$ are connected as shown in the figure. What is the obtuse angle between the chords AE and CG at the point X?



ANSWER 132°

SOLUTION

The calculation is made easier because the chords AE and CG have the same length. Mark in the centre O of the circle passing through all the vertices of the polygon. We can see that O, X and D lie on a straight line by the symmetry of the figure about the line OD. It follows that the obtuse angle $\angle AXG$ required is twice $\angle AXO$.



We can determine $\angle AXO$ by finding $\angle AOX = \angle AOD$ and $\angle OAX = \angle OAE$, and then using the angle sum in $\triangle AXO$.

Now $\angle AOD = \frac{3}{15} \times 360° = 72°$.

Similarly $\angle AOE = \frac{4}{15} \times 360° = 96°$. Since $\triangle OAE$ is isosceles with $OA = OE$, it follows that $\angle OAE = \frac{1}{2}(180° - 96°) = 42°$.

Then using the angle sum in $\triangle AXO$, we find $\angle AXO = 180° - 72° - 42° = 66°$, and so $\angle AXG = 2 \times 66° = 132°$.

3. An athlete walks d km at 5 km per hour and then runs another d km at 12 km per hour. Find the average speed of the athlete over the whole $2d$ km journey.

Answer $7\frac{1}{17}$ km/h

Solution

The time taken to travel the first section of the journey is $\frac{d}{5}$ hours. Similarly, the second time is $\frac{d}{12}$ hours. Hence the total time in hours for the complete journey is

$$\frac{d}{5} + \frac{d}{12} = \frac{12d + 5d}{60} = \frac{17}{60}d.$$

The total journey distance is $2d$ km, so the average speed in km/h is

$$\frac{2d}{\frac{17}{60}d} = \frac{2}{\left(\frac{17}{60}\right)} = \frac{120}{17} = 7\tfrac{1}{17}.$$

> The calculation could be done by assigning a number to d as d cancels out after the third line. You met a similar idea in question 1.
>
> Also, note the use of the parentheses around $\frac{17}{60}$ in the final calculation; this is to make it unambiguous what is the numerator and what is the denominator of the big fraction.

4. What is the least positive integer n for which the number $2003n$ ends with ... 113 when written in decimal notation?

Answer 371

Solution

The last three digits of the product $2003n$ are only affected by the last three digits of 2003 and the last three digits of n.

The last three digits of 2003 are just 003 or 3, and let the last three digits of n be abc_{10}, where we write the small '10' to indicate that a is the hundreds digit, b is the tens digit and c is the ones (units) digit.

We then need $3 \times abc_{10} = \ldots 113$.

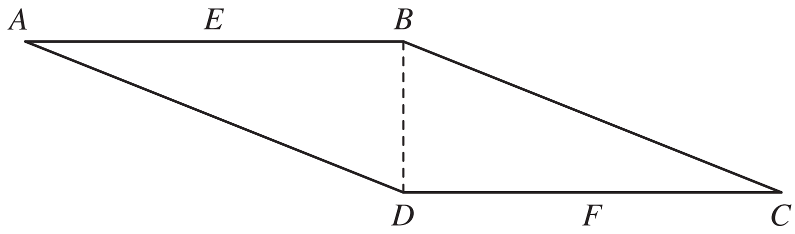
Considering the ones (units) digit, $c = 1$.

Considering the tens digit, $b = 7$, and there is a carry of 2 to the hundreds column.

Considering the hundreds digit, $3 \times a + 2 = \ldots 1$, so $a = 3$.

Thus $n = \ldots 371$, and the smallest positive integer n with this property is 371.

5. In the parallelogram $ABCD$ shown below, $AB = 5$, $BD = 2$ and $\angle ABD$ is a right angle. E and F are the midpoints of AB and CD. AF meets DE at G. BF meets CE at H. Find the area of the quadrilateral $EHFG$.

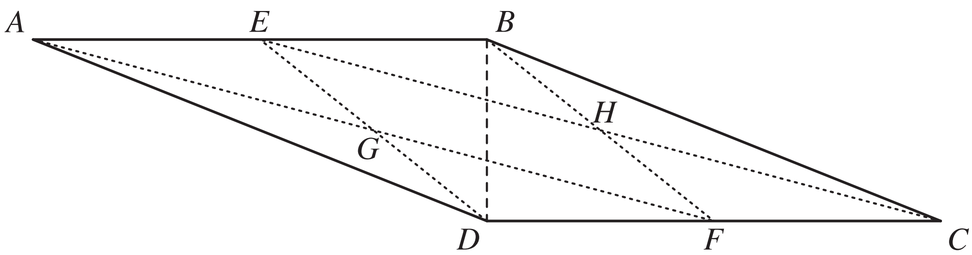


Answer $\frac{5}{2}$

Solution

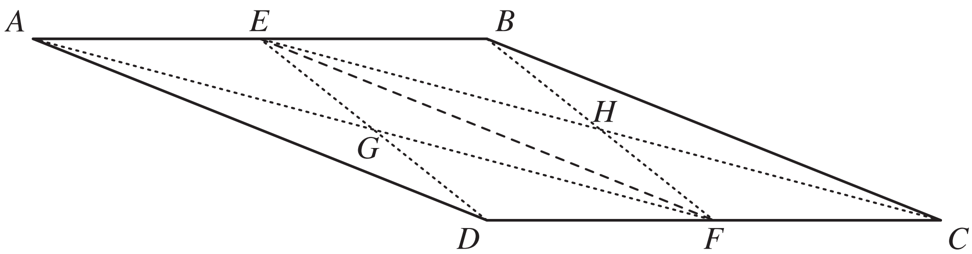
The area of the parallelogram $ABCD$ is $5 \times 2 = 10$.

This is a drawing of all of the lines described in the question:



Note that AF and EC are parallel, as $AE = FC$ and they are parallel, which makes $AFCE$ a parallelogram. Similarly, $BFDE$ is a parallelogram and BF is parallel to ED.

This shows that the diagram has rotational symmetry; if we draw in the line EF joining the midpoints of AB and DC, which is parallel to AD and BC, this will become even clearer:



We now see that $AEFD$ and $EBCF$ are congruent parallelograms. As AF and ED are the diagonals of $AEFD$, G is the midpoint of these diagonals. Hence $\triangle AEG$ has the same area as $\triangle AGD$ since both have equal length bases EG and GD and the same perpendicular height from A onto the line DE. Similarly, all four of the triangles that $AEFD$ is composed of have equal areas, and likewise for $EBCF$. We therefore have 8 triangles of equal area, two of which make up the parallelogram $EGFH$, so this parallelogram has area $\frac{10}{4} = \frac{5}{2}$.

Note that $\angle ABD$ being a right angle was only used to find the area of $ABCD$; that $EHFG$ has area one-quarter of $ABCD$ is true for any parallelogram $ABCD$.

6. There are n pirates in the crew of a pirate's ship. One day they capture a chest with m gold coins and begin to share the booty. They split the coins into n heaps, one for each pirate. On counting their heaps, the captain finds that he has 2020 coins but the botswain has only 909 coins. They decide to re-apportion the coins starting with the current heaps. On each round, the pirate currently having the smallest number of coins takes one coin from each of the other heaps. (If there are two or more heaps which have an equally small number of coins, they decide which pirate takes the coins by tossing a coin.)

(a) Suppose $n = 10$ and m is a multiple of 10. Can the pirates equalise the number of coins in each heap using such transfers?

(b) Now suppose $n < 100$ and m is not given. After some transfers, all the heaps contain the same number of coins. How many pirates are there in the crew?

Answer 11

Solution

(a) On any round, the smallest heap will gain 9 coins while every other heap loses 1 coin. The difference between any two heaps, neither of which are the smallest, will not change. The difference between the size of the heap which was smallest and any other heap will reduce by 10. But the difference between the captain's heap and the botswain's heap was originally 1111, so it can never be reduced to zero no matter how many rounds are performed.

We should actually be a little more careful, as if one heap has 15 coins and the smallest one has 12 coins, then after a round, these heaps become 14 and 21 respectively, so the absolute difference in size has gone from 3 to 7. So instead we should specify that we are always going to subtract the size of one heap from the other one; in this case, the difference will go from 3 to -7 or from -3 to 7. So the difference between the smallest heap and another heap will always change by 10, but it might be $+10$ or -10 depending on the order in which we perform the subtraction. So the difference in size of two heaps will always change by a multiple of 10. Since 1111 is not a multiple of 10, the difference between the captain's heap and the botswain's heap can never be reduced to zero.

(b) Using the same idea but replacing 10 by n shows that the difference between the sizes of two heaps after a round will either stay the same or will reduce by n. A difference can only reduce to zero if n divides into 1111. Now 1111 has prime factorisation 11×101. This leaves only 11, 101 or 1111 as possible values for n. But we are given that $n < 100$, so $n = 11$.

7. Show that given five distinct integers, one can always choose three of them whose sum is divisible by 3.

Solution

Consider the remainder when an integer is divided by 3. It can be 0 (for a multiple of 3), 1 or 2. There are three general cases for the possible remainders of the five integers.

(a) All three types of remainder appear among the remainders. If we then choose one of each type, the sum of these will be a multiple of 3, since the remainders sum to 3.

(b) If there is only one type of remainder, then choosing any three will produce a sum which is a multiple of 3, since the remainders sum to 0, 3 or 6.

(c) If there are just two types of remainder, then at least three of them will be of one type. Choose three of those, and they sum to a multiple of 3 as in the previous case.

We can also describe these algebraically: a remainder of 0 means that the integer can be written as $3a$, where a is an integer; a remainder of 1 gives $3a + 1$ and a remainder of 2 gives $3a + 2$. Adding three expressions like these in the ways described will always give an expression which is a multiple of 3.

8. Each cell of an 8 by 8 chessboard has a 0 or a 1 written in it. Prove that if we work out the sums of the numbers in each row, in each column and in each of the two diagonals, we will find that at least three of these sums are equal.

Solution

There are 18 sums calculated (8 rows, 8 columns and 2 diagonals). Each of these sums takes one of the 9 values between 0 and 8. If one of these 9 values occurs only once or not at all, then there must be a value that occurs more than twice. Thus the only way that result would not be true is if there were a way to fill the chessboard so that there were 2 sums with value 0, 2 sums with value 1, etc. Let us suppose that it is possible to fill the board in such a way. By showing that this is impossible, we will be done.

If a sum of 8 appears in a diagonal, then none of the rows or columns will sum to 0, so there is at most one sum of 0 (the other diagonal). If a sum of 8 appears in both a row and a column, then there cannot be any sums of 0, so this is not possible. So the two sums of 8 must appear in two rows or two columns; we may suppose without loss of generality that they appear in two rows. (If they appear in two columns instead, we can just rotate the board by 90° without changing any of the sums.) The two sums of 0 must then appear in two other rows. Now no diagonal or column can have a sum of 1 or of 7, so these sums must account for the remaining four rows.

Within the four rows with sum 0 or sum 1, there are just two 1's (the other 30 entries being 0's). Similarly, in the four rows with sum 7 or sum 8, there are just two 0's (the other entries being 1's). These four entries (the two 1's and the two 0's) can be placed in a maximum of four columns. Hence there are at least four columns with four 1's and four 0's, and these columns all have a sum of 4.

This shows that it is not possible to fill the board in this way, and we are done.

United Kingdom
Mathematics Trust

Mentoring Scheme

Supported by OxFORD ASSET MANAGEMENT

Hypatia

Sheet 6

Solutions and comments

This programme of the Mentoring Scheme is named after Hypatia of Alexandria (c. 370–415 CE).

See http://www-history.mcs.st-and.ac.uk/Biographies/Hypatia.html for more information.

Enquiries about the Mentoring Scheme should be sent to:

Mentoring Scheme, UK Mathematics Trust, School of Mathematics, University of Leeds, Leeds LS2 9JT

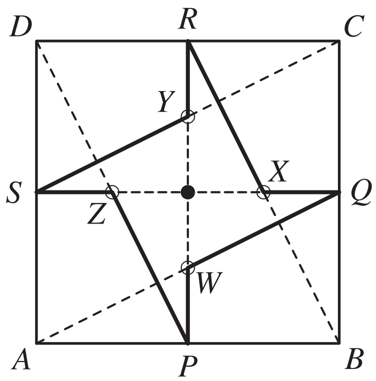
☎ 0113 343 2339 mentoring@ukmt.org.uk www.ukmt.org.uk

Version 1.1, Mar 2020

1. $ABCD$ is a square with side length 2. P, Q, R and S are the midpoints of AB, BC, CD and DA respectively. W is where AQ meets PR. Similarly X, Y and Z are the intersections of BR with QS, CS with RP, and DP with SQ respectively. What is the area of the octagon $PWQXRYSZ$?

Answer 1

Solution



The diagram shows that the octagon is made up of four congruent triangles, all meeting at the centre of the square.

Letting O be the centre of the square, consider $\triangle OSY$. This is similar to $\triangle QSC$ and has half the side length. The base and height of $\triangle QSC$ are 2 and 1, so the base and height of $\triangle OSY$ are 1 and $\frac{1}{2}$, giving it area $\frac{1}{4}$. The four triangles together thus have total area 1.

2. If n is a positive integer, how many different values are possible for the remainder when n^2 is divided by 7?

Answer 4

Solution

We calculate the remainders for small values of n when n^2 is divided by 7:

n	n^2	remainder
1	1	1
2	4	4
3	9	2
4	16	2
5	25	4
6	36	1
7	49	0
8	64	1

After this we might suspect that there is a sequence 1, 4, 2, 2, 4, 1, 0, 1, 4, 2, 2, 4, . . . for the remainders. To prove we are right, let q be an integer and $n = 7q + r$, where $0 \leq r < 7$; then q and r are the quotient and remainder when n is divided by 7. We can now calculate

$$\begin{aligned} n^2 &= (7k + r)^2 \\ &= (7k + r)(7k + r) \\ &= 49k^2 + 7kr + 7kr + r^2 \\ &= 7(k^2 + 2kr) + r^2 \\ &= \text{(a multiple of 7)} + r^2 \end{aligned}$$

Now we see that it really was only necessary to consider n between 1 to 7 inclusive (or equivalently between 0 and 6 inclusive, even though the question specified positive integers).

3. Zara has two piles of counters, one with 233 counters and the other with 144 counters. She starts a process whereby at each move she takes away from the larger pile a number of counters equal to the number in the smaller pile, and discards these counters in a bin. How many moves does she make before one of the piles disappears?

What can you conclude about adding up the numbers in the Fibonacci sequence?

Answer 12

Solution

After each move, the piles have numbers of counters as represented by neighbouring pairs in this sequence: 233, 144, 89, 55, 34, 21, 13, 8, 5, 3, 2, 1, 1, 0, so on the 12th move, one of the piles disappears.

This is the start of the Fibonacci sequence written in reverse. We see that we have discarded all these counters into the bin. Thus all but 1 of the $233 + 144 = 377$ counters are in the bin.

Thus $1 + 1 + 2 + 3 + \cdots + 144 = 376$, which is one less than the next Fibonacci number. In general, the sum of the first n Fibonacci numbers is one less than the $(n + 2)$th Fibonacci numbers. In symbols, if we write $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, and so on, we have

$$F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1.$$

4. Fred added up all the positive integers from 1 to some number n on his calculator and obtained a total of 2020. By mistake he had entered one number twice. Find the number he entered twice.

Answer 4

Solution

If you started from the hint, you will have found that the sum of the integers from 1 to 9 is half of 9×10.

If you look at the online notes or back to Hypatia sheet 1 question 8, you will see that we can derive the formula $\frac{1}{2}n(n + 1)$ for the sum of the integers from 1 to n.

We therefore want to find an n which gives a sum of just over 2020. To do this, we could try to solve the equation $\frac{1}{2}n(n + 1) = 2020$ but this is tricky.

Alternatively, we could make an estimate by solving $\frac{1}{2}n^2 = 2020$, so $n^2 = 4040$. This gives n between 63 and 64. Now if $n = 63$, we have $\frac{1}{2} \times 63 \times (63 + 1) = 2016$, so the extra number added by mistake is 4.

If n were 62, we would have $\frac{1}{2} \times 62 \times 63 = 1953$, and the extra number added would be 67, which is bigger than 62, so this is not possible. It is even worse if $n < 62$.

Likewise, if n were 64, the sum would be 2080, which is too large.

5. In a rectangular room the walls face north, east, south and west, and there are four doors. Three people in the room make statements about the positions of the doors. Andrew says, "There are no doors in the south wall." Beth says, "There are only doors in the north wall." Cally says, "In each wall there is at most one door." How much can you deduce from this?

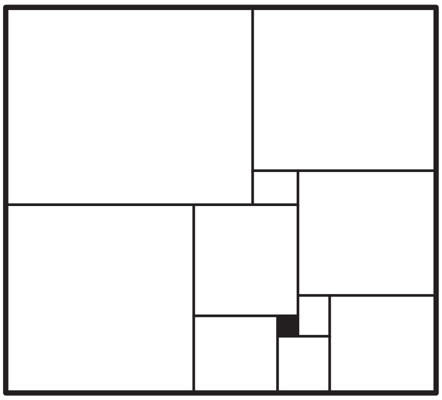
Solution

One good way of sorting out this sort of logic is to make out a truth table. In this table, T represents a person telling the truth and F a person telling a lie (the opposite to the truth). NP represents the conclusion that this combination of truth and lies is Not Possible because the statements contradict each other.

Andrew	Beth	Cally	conclusion
T	T	T	NP
T	T	F	4 doors in N wall
T	F	T	NP
T	F	F	no doors in S wall
F	T	T	NP
F	T	F	NP
F	F	T	one door in each wall
F	F	F	$\geq$ 1 door in S wall, some wall has no door

We can also deduce that if Cally is telling the truth, the other two are lying. If Andrew is telling the truth, then Cally is lying, while if Beth is telling the truth, so is Andrew.

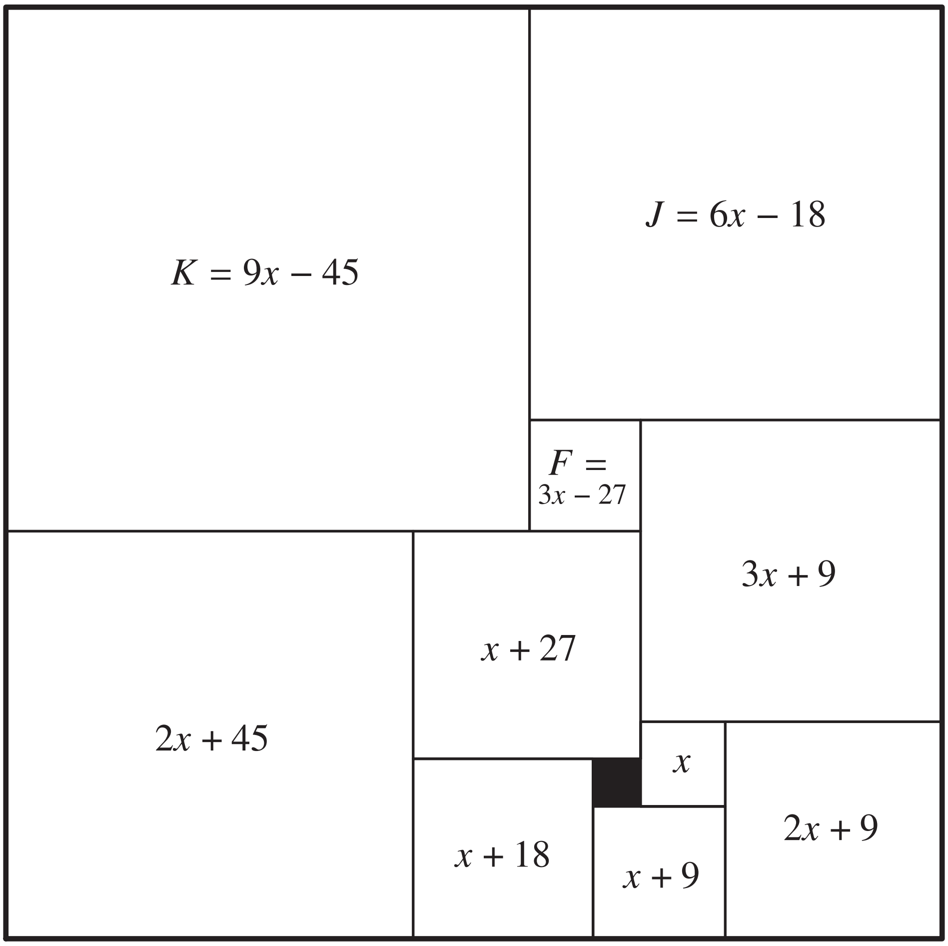
6. The rectangle shown is dissected into 11 *squares* of different sizes. (Do not try to measure from the diagram because the diagram is not accurately drawn.) The smallest square, shown black, is 9 by 9. What are the dimensions of the rectangle?



Answer 177 width by 176 height

Solution

Let the side length of the square to the right of the black square (likely the next smallest square) be x. We can then fill in the side lengths of the squares working outwards. They are shown in the middle of each square on this diagram (which is accurately drawn, though it did not need to be).



After working out all but squares F, J and K, we can see that the height of the bottom right two squares together is $5x + 18$ while the height of the two squares to the left and around the black square together is $2x + 45$, which is the same as the height of the bottom-left square. This means the height of square F (so far not calculated) is $3x - 27$. We now work out that square J has base $6x - 18$ and square K has height $9x - 45$.

The top two squares together must have the same width as the bottom four squares together so

$$\begin{aligned} & (9x - 45) + (6x - 18) = (2x + 45) + (x + 18) + (x + 9) + (2x + 9) \\ \Longrightarrow \quad & 15x - 63 = 6x + 81 \\ \Longrightarrow \quad & 9x = 144 \\ \Longrightarrow \quad & x = 16 \end{aligned}$$

Thus the width of the whole rectangle is $15x - 63 = 15 \times 16 - 63 = 240 - 63 = 177$, and its height is the sum of the heights of the two squares on the left, $11x = 176$.

> This was the first squared rectangle found by A. H. Stone, a brief biography of whom can be found at http://www.squaring.net/history_theory/brooks_smith_stone_tutte.html. The story of squaring the square can be found in Martin Gardner, "Origami, Eleusis, and the Soma Cube" (MAA, 2008). Stone found this squared rectangle without knowing that the side length of the smallest square was 9; he called it x and the length of the next smallest square y.

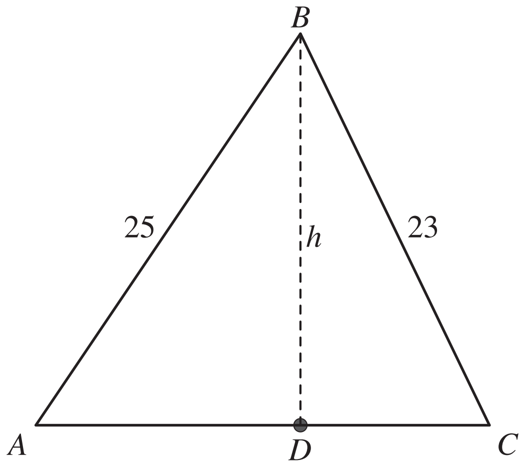
7. In $\triangle ABC$, $AB = 25$, $BC = 23$ and $AC = 24$. A perpendicular is drawn from B to AC meeting AC at D. Find the value of $AD - DC$.

Note: You will find the identity $x^2 - y^2 = (x - y)(x + y)$ useful.

[Note that Question 8 and the hints are on the next page]

Answer 4

Solution



We can write down two equations from Pythagoras's Theorem:

$$25^2 = AD^2 + h^2$$
$$23^2 = CD^2 + h^2$$

and we can subtract them to give

$$AD^2 - CD^2 = 25^2 - 23^2.$$

If we think of AD as x and CD as y in the identity suggested, then we obtain:

$$\begin{aligned} & (AD - CD)(AD + CD) = 625 - 529 \\ \implies & \quad (AD - DC) \times 24 = 96 \\ \implies & \quad AD - DC = 4 \end{aligned}$$

It is perhaps easier to do the working by putting $AD = x$ and $CD = y$ at the start. Any other method will lead to finding that h is not rational.

8. Let $N = 2001 \times 2002 \times 2003 \times 2004 \times 2005 \times 2006 \times 2007 \times 2008 \times 2009 \times 2010 \times 2011$ and let M^3 be the maximum cube number that is a factor of N. What is the value of M?

Answer 84

Solution

One approach is to factorise each of the numbers from 2001 to 2011, but that seems quite tricky if there are no obvious small factors. A more straightforward approach to to test potential prime factors of N instead, and this is the approach we will take. We must test both the primes *and* their powers for divisibility.

- 2 divides into (is a factor of) 2002, 2004, 2006, 2008 and 2010. 4 divides into 2004 and 2008. 8 divides in 2008. 16 does not divide into any of them. Hence 2^8 divides into N (but 2^9 does not).
- 3 divides into 2001, 2004, 2007 and 2010. 9 divides into 2007. 27 does not divide into any of them. Hence 3^5 divides into N.
- 5 divides into 2005 and 2010. 25 does not divide into either of them. Hence 5^2 divides into N.
- 7 divides into 2002 and 2009. 49 divides into 2009. 343 does not divide into either of them. Hence 7^3 divides into N.
- 11 divides into 2001 but 11^2 does not.
- Each prime $p \geq 13$ can divide into at most one of $2000, \ldots, 2011$. Since $13^3 = 2197 > 2011$, p^3 will not divide into N for any $p \geq 13$.

If M^3 divides into N, and M is even, then M^3 has to be divisible by $2^3, 2^6, 2^9, \ldots$. Although 2^8 is a factor of N, we can only use 2^6 in M^3 because the index must be a multiple of 3. In the same way, we can use 3^3 but not 3^5, and 7^3 but not any of the other primes. Thus

$$\begin{aligned} M^3 &= 2^6 \times 3^3 \times 7^3 \\ \text{so} \quad M &= 2^2 \times 3 \times 7. \end{aligned}$$

United Kingdom
Mathematics Trust

Mentoring Scheme

Supported by OxFORD ASSET MANAGEMENT

Hypatia

Sheet 7

Solutions and comments

This programme of the Mentoring Scheme is named after Hypatia of Alexandria (c. 370–415 CE).

See http://www-history.mcs.st-and.ac.uk/Biographies/Hypatia.html for more information.

Enquiries about the Mentoring Scheme should be sent to:

Mentoring Scheme, UK Mathematics Trust, School of Mathematics, University of Leeds, Leeds LS2 9JT

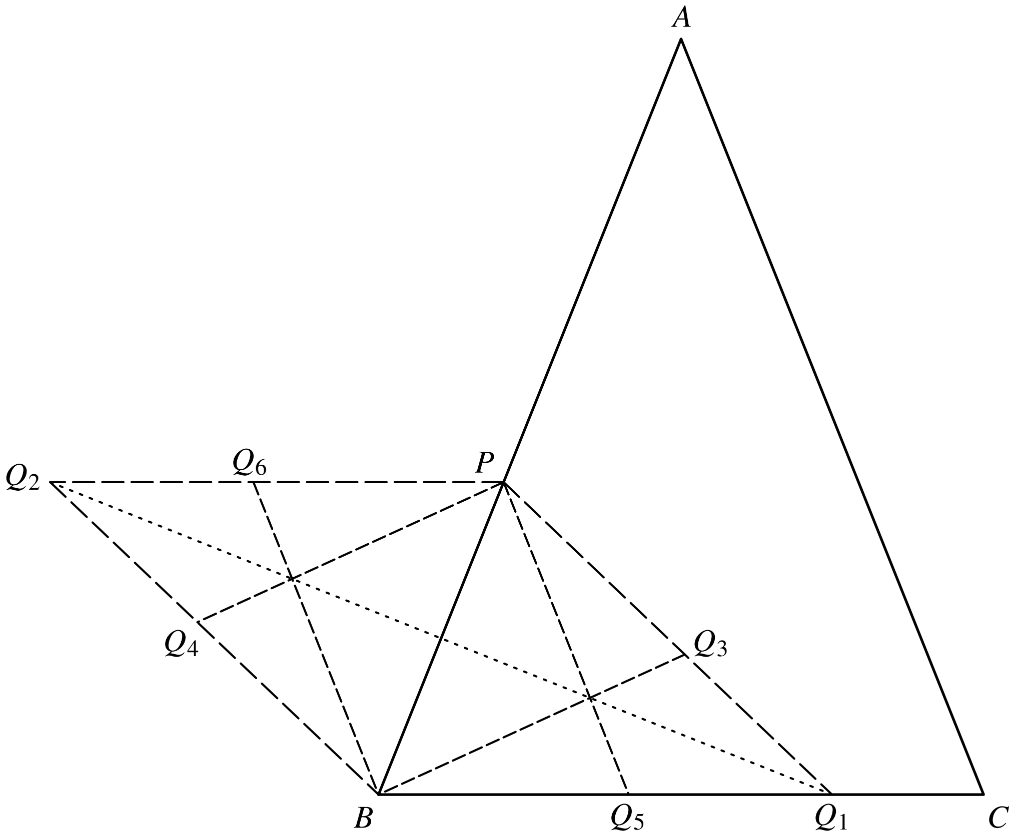
☎ 0113 343 2339   🖷 0113 343 5500   mentoring@ukmt.org.uk   www.ukmt.org.uk

1. $\triangle ABC$ is isosceles with $AB = AC \neq BC$. P is chosen anywhere on AB. How many positions can the point Q take (inside or outside the triangle) so that $\triangle BPQ$ is similar to $\triangle ABC$? The letters B, P and Q can be considered to correspond to A, B and C in any order. You should include a well-drawn diagram in your solution.

Answer 6

Solution

The various configurations are shown in the diagram.



You will see that there are various symmetries in the overall figure around BP and Q_1Q_2.

2. An escalator goes up from the first to the second floor of a department store. Dion walks up it at a constant pace. Raymond tries to walk down from the top at the same pace as Dion. (Their pace is measured relative to the escalator.) Dion arrives at the second floor after 12 steps. Raymond arrives at the first floor after 60 steps.
How many steps would it take Dion to get upstairs if the escalator were stopped?

Answer 20

Solution

(a) Change the problem slightly. Imagine there are three escalators, one going up, one stopped and one going down and that Raymond takes the *down* escalator to go *up*. When Dion reaches the top, Raymond has only gone $\frac{1}{5}$ of the way up his escalator. Hence someone going up the stationary escalator at the same pace will have climbed $\frac{3}{5}$ of the way, that is, situated half way between Dion at the top and Raymond $\frac{1}{5}$ of the way up. Dion will need $\frac{5}{3} \times 12 = 20$ steps to reach the top if the escalator is stopped.

(b) Alternatively define a unit of time as that needed for Dion and Raymond to take a step, and let the escalator travel v steps in this unit of time. Then Dion takes 12 units of time to travel a distance of $12 + 12v$ steps, while Raymond takes 60 units of time to travel a distance of $60 - 60v$ steps. These are equal distances so

$$\begin{aligned} 12 + 12v &= 60 - 60v \\ 72v &= 48 \\ v &= \tfrac{2}{3} \end{aligned}$$

The total distance travelled is $12 + 12 \times \frac{2}{3} = 20$ steps.

(c) Here is an alternative algebraic solution. Suppose the escalator travels at s steps per minute, that there are n steps to be climbed when the escalator is stationary and that both Dion and Raymond take r steps per minute while on the escalator. The time taken by Dion to travel between the floors is:

$$T_d = \frac{n}{r+s}$$

while the time taken by Raymond to travel between the floors is:

$$T_r = \frac{n}{r-s}$$

where we assume that $r > s$ otherwise Raymond will go backwards.
As Dion and Raymond take 12 and 60 steps respectively, $5T_d = T_r$. Hence:

$$\begin{aligned} \frac{5n}{r+s} &= \frac{n}{r-s} \\ 5(r-s) &= r+s \\ 5r - 5s &= r + s \\ 4r &= 6s \\ r : s &= 3 : 2 \text{ or both move at } 1\tfrac{1}{2} \text{ times the speed of the elevator.} \end{aligned}$$

Since Dion takes 12 steps to get to the top, the elevator has moved 8 steps in that time and Dion would therefore have taken 20 steps to the top if the escalator had been stationary.

3. This question is set with the end of the year 2018 in mind.
(i) Which values from 1 to 10 can you form with the digits of 2018, each digit being used exactly once in the order in which it appears? You are allowed to use $+ - \times \div ()$, or to put digits together in order to form the values. For example, $-20 \times 1 + 8$ would be allowed for -12.
(ii) Can you obtain all the values from 1 to 10 using the digits in order if you are also allowed to use powers, square roots and factorials. How?
Note that $1! = 1, 2! = 2 \times 1, 3! = 3 \times 2 \times 1, 4! = 4 \times 3 \times 2 \times 1$, etc. It is *defined* that $0! = 1$. This last result follows from the rules of factorials by a method similar to that showing $5^0 = 1$ follows from the rules of indices.

Answer (i) 2, 5, 6, 7, 8, 9, 10

Solution

What follows is a list of one possibility for each value. Examples using powers, square roots and factorials are shown. There are many more possibilities.

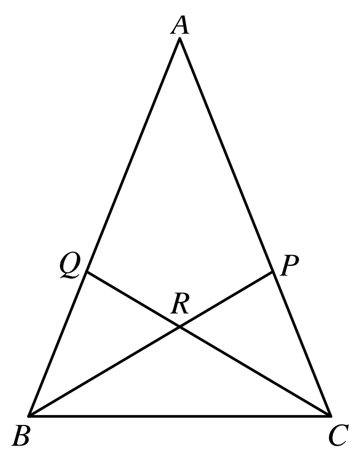
$$
\begin{aligned}
1 &= 2 \times 0 + 1^8 \\
2 &= 20 - 18 = 2 \\
3 &= (2 + 0! + 1)! \div 8 \\
4 &= \sqrt{(2 + 0 \times 1) \times 8} \\
5 &= -2 + 0 - 1 + 8 \\
6 &= -2 + 0 \times 1 + 8 \\
7 &= -2 + 0 + 1 + 8 \\
8 &= 2 \times 0 + 1 \times 8 \\
9 &= 2 \times 0 + 1 + 8 \\
10 &= 2 + 0 \times 1 + 8
\end{aligned}
$$

4. $\triangle ABC$ is isosceles with $AB = AC$. $\angle BAC < 60°$. P is on AC and Q is on AB such that $BP = CQ = BC$. R is the intersection of BP and CQ. $BR = BQ$ and $CR = CP$.
What is the magnitude of $\angle BAC$?

Answer 36°

Solution

This diagram is realistic but not exactly to scale.



Let $\angle ABC = x$. Since $AB = AC$, $\angle ACB = x$. We are given $BP = CQ = BC$ so $\triangle BCP$ and $\triangle CBQ$ are isosceles. Hence $\angle BPC = \angle CQB = x$. From the angle sum of triangle we can show that $\angle BAC = \angle CBP = \angle BCQ = 180^\circ - 2x$.

We are also given that $CR = CP$ and $BR = BQ$ so $\triangle CPR$ and $\triangle BQR$ are isosceles. Hence $\angle CRP = \angle BRQ = x$ and $\angle PCR = \angle QBR = 180^\circ - 2x$.

By angle sum of triangle $\angle BAC + \angle PCR + \angle BCQ + \angle CBP + \angle QBR = 180^\circ$. As these angles are all equal, they must all be 36°.

Alternatively for the final step, $\angle PCR + \angle QCB = \angle ACB$, and so $(180 - 2x) + (180 - 2x) = x$ leads to the same result.

> If you look carefully at a regular pentagon and its diagonals, you will be able to pick out this configuration of lines.

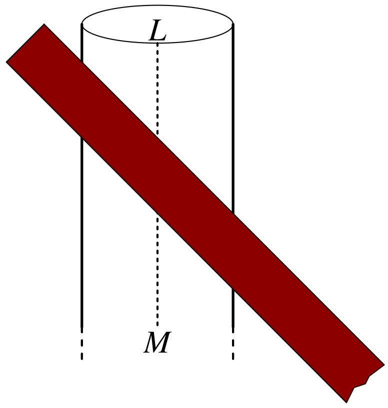
5. Martha is about to colour the vertices of a square grid forming 100 squares of unit length sides using just three colours. Before Martha begins, Daniella accepts the challenge of drawing a circle with its centre on the vertex at the centre of the square grid in such a way that it is large enough for Daniella to be certain that it will contain at least three vertices of the same colour, no matter how Martha decides to colour the vertices. (The vertices counted may lie on the circumference of the circle or inside it.) What is the radius of the smallest such circle that Daniella can draw?
Give an exact value, not an approximate decimal value.

Answer $\sqrt{2}$

Solution

The required circle passes through the corners of a 2 by 2 square encompassing 9 vertices. If it were slightly smaller, then the circle would only encompass 5 vertices and Martha could place 2 of two colours and 1 of the other on these vertices.

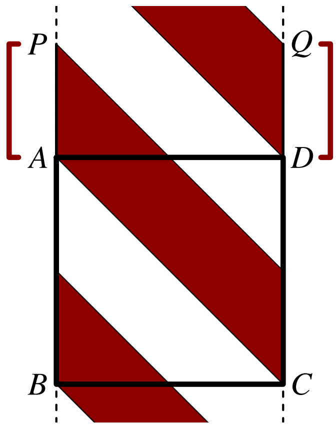
6. We wind a length of sticky plastic red tape round a cylindrical white pole which has diameter 4 cm. The tape makes an angle of 45° with the dotted line LM down the pole (parallel to the axis of the cylinder) as shown in the figure. In this way we obtain two spirals down the pole, one red and one white. These spirals have the same width. What is the width of the plastic tape?



Answer $\pi\sqrt{2}\,cm$

Solution

Follow the hint given. On the flattened diagram below, the vertical edges represent the two sides of the cut LM. The tape spirals down from the top to disappear between D and Q on the right and reappear between P and A on the left.



One complete turn of the tape leads us to examine $ABCD$ which is a square because the tape runs diagonally across at 45°. The side AD of this square is the circumference of the pole, that is, 4π. The width of the tape is measured in the direction of the diagonal BD. Using Pythagoras's theorem, we have:

$$BD^2 = (4\pi)^2 + (4\pi)^2 = (4\pi)^2 \times 2$$

Hence $BD = 4\pi\sqrt{2}$. We can now see that the tape crosses this diagonal twice and so does the exposed white space. We have to divide the length of BD by 4 to get the width of the tape.

Another starting point to a solution is based on measuring diagonally across the tape.

7. A palindrome is a word or number that reads the same backwards as forwards, e.g. LEVEL or 9889. Find the sum of all the four digit positive integer palindromes.

ANSWER 495000

SOLUTION

In the palindrome $yxxy_{10}$ we can choose the x and y *independently*. This means that for each y ranging from 1 to 9, we can have every value of x from 0 to 9. Likewise, for each x ranging from 0 to 9, we can have every value of y from 1 to 9. Note that y can not be 0.
Thus in each of the numbers $1xx1_{10}, 2xx2_{10}, \ldots, 9xx9_{10}$ the xs will add to $100 \times (0 + 1 + \cdots + 9) + 10 \times (0 + 1 + \cdots + 9) = 4500 + 450 = 4950$. Since there are nine cases of y, the total of the xs in the tens and hundreds positions in all the palidromes will be $9 \times 4950 = 44550$.
$1001 + 2002 + \cdots + 9009 = 45045$ and each case occurs 10 times. So they contribute 450450 to the total.
Finally $450450 + 44550 = 495000$.

8. Determine whether 25^{24} can be written as the sum of two positive squares. Can you deduce whether or not 75^{24} could be written as the sum of two positive squares?

ANSWER Both are possible

SOLUTION

(a) $25 = 5^2$. Hence $25^{24} = 5^{48}$. We know that $5^2 = 3^2 + 4^2$. We can scale up this relationship to give $m(5)^2 = m(3)^2 + m(4)^2$ then find the value of m which makes it work. We obtain what we require by choosing $m = 5^{46}$ which in itself is a square so the sum is possible.

The squares are $(3 \times 5^{23})^2$ and $(4 \times 5^{23})^2$.

We might also note that 25^2 can be written as both $25 \times 3^2 + 25 \times 4^2$ and $7^2 + 24^2$, which gives another way of writing 25^{24} as a sum of two positive squares, namely $(5^{22} \times 7)^2 + (5^{22} \times 24)^2$.

(b) For each way of writing 25^{24} as a sum of two squares, we can multiply each square by 3^{24} which is itself a square, thereby obtaining a way of writing 75^{24} as a sum of two squares. For example, $75^{24} = (3^{12} \times 3 \times 5^{23})^2 + (3^{12} \times 4 \times 5^{23})^2$.

It turns out that 25^3 can also be written as $44^2 + 117^2$ which gives another way of writing 25^{24}. In total, there are 24 different ways to write 25^{24} as a sum of two squares. 75^{24} can be likewise be written as the sum of two positive squares in 24 different ways. It turns out that every way that 75^{24} can be written as a sum of two squares arises this way - there is no way to write 75^{24} as a sum of two squares which is not of the form $(3^{12}x)^2 + (3^{12}y)^2$ for some x and y.

United Kingdom
Mathematics Trust

Mentoring Scheme

Supported by OxFORD ASSET MANAGEMENT

Hypatia

Sheet 8

Solutions and comments

This programme of the Mentoring Scheme is named after Hypatia of Alexandria (c. 370–415 CE).

See http://www-history.mcs.st-and.ac.uk/Biographies/Hypatia.html for more information.

Enquiries about the Mentoring Scheme should be sent to:

Mentoring Scheme, UK Mathematics Trust, School of Mathematics,
University of Leeds, Leeds LS2 9JT

☎ 0113 343 2339 mentoring@ukmt.org.uk www.ukmt.org.uk

Version 1.0, January 2019

1. You choose a three digit number a such that when the digits of a are reversed, they form a *different* three digit number b. Let c be the (non-negative) difference between a and b. How many possible values of c are there?

ANSWER 8 values

SOLUTION

Let the digits of the larger of a and b be x, y and z. Then the larger of a and b is $100x + 10y + z$ and the smaller is $100z + 10y + x$. We simplify $a - b$:

$$(100x + 10y + z) - (100z + 10y + x) = 100(x - z) + (z - x) = 99(x - z).$$

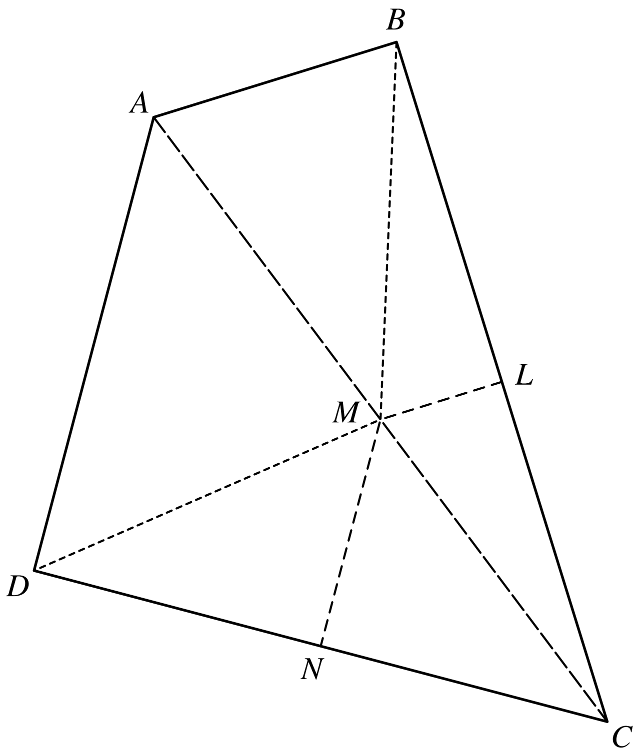
As 100x+10y+z is the larger number, we have $1 \leq z < x \leq 9$.
The 8 possible values of c are therefore 99, 198, 297, 396, 495, 594, 693, 792, 891.

> It is worth mentioning a similar party trick. Take a three digit number c (with a 0 in the hundreds column if necessary, reverse the digits to give d and work out $c + d$. It always comes to 1089.

2. $ABCD$ is a quadrilateral such that $\angle ABC = \angle CDA = 90°$ and $\angle BCD = 58°$. L, M and N are the midpoints of BC, AC and DC respectively. What are the magnitudes of $\angle LMN$ and $\angle BMD$?

ANSWER 122 and 116

SOLUTION



Since L is the midpoint of BC and M is the midpoint of AC, then LM is parallel to BA. Hence $\triangle CLM$ is similar to $\triangle CBA$ so we can deduce that $\angle CLM = 90°$. (Deduction 1)

Similarly we can show that $\angle CNM = 90°$. Hence in quadrilateral $CLMN$

$$\angle LMN + 90° + 58° + 90° = 360°$$

from which we deduce that $\angle LMN = 122°$.

(You might notice that $MLCN$ is an enlargement of $ABCD$ with centre C by scale factor $\frac{1}{2}$.)

Deduction 1 shows that $\triangle CLM$ is congruent to $\triangle BLM$ (using two sides and an included angle, that is, "SAS") so $\angle CML = \angle BML$. Similarly, $\angle CMN = \angle DMN$. Hence

$$\angle CML + \angle CMN = \angle BML + \angle DMN$$

so $\angle BML + \angle DMN = 122°$. Round the point M

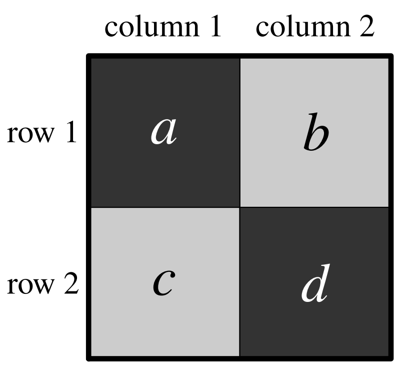
$$\angle BMD + (\angle BML + \angle DMN) + \angle LMN = 360°$$

so $\angle BMD = 360° - 2 \times 122° = 116°$.

> **3.** Each square of a standard chess board with 8 rows and 8 columns is marked with an integers. The rows and the columns of the chessboard are numbered from 1 to 8. The square in row 1 column 1 is black. The sum of the numbers on the white squares is 28 and the sum of the numbers in the odd-numbered columns is 47. If we change the sign of all the numbers on the white squares, what is the sum of the numbers on the odd-numbered rows?

ANSWER 19

SOLUTION



It is easier to think of what happens in a 2 by 2 block as shown in the diagram because we can extend our reasoning to sixteen such blocks put together to form the 8 by 8 chess board. We note that $a + c = 47$ (first column) and $b + c = 28$ (white squares). Subtracting the two equations we obtain $a - b = 19$. This can be written $a + (-b) = 19$ which is the sum of the odd-numbered row with the white numbers changed in sign. We can now argue that a represents the *sum* of all the numbers in cells which are in both odd-numbered rows and odd-numbered columns. A similar idea can be applied to the other letters.

4. With real numbers, it is always true that

$$p \times (q + r) = (p \times q) + (p \times r).$$

(This is known as the distributive law.)

Show that it is only true that

$$p + (q \times r) = (p + q) \times (p + r)$$

if $p + q + r = 1$ or $p = 0$ (or both).

Answer 4

Solution

$$\begin{aligned}(p + q) \times (p + r) &= p \times p + p \times r + q \times p + q \times r \\ &= p^2 + pr + qp + qr \\ &= p(p + r + q) + qr\end{aligned}$$

Now we can see how to match this up with $p + (q \times r) = p + qr$. If these are equal then $p(p + q + r) = p$ which we can rearrange to obtain

$$p((p + q + r - 1) = 0.$$

Since a produce is zero if and only if at least one of the two factors is zero, then the original expressions are equal (if and) only if $p = 0$ or $p + r + q = 1$.

5. Alice has a pack of 52 playing cards from which she removes all the picture cards (Jacks, Queens and Kings). Ace takes value 1, the rest take the value shown on the card. She shuffles the pack and hands it to Bob face down, instructing Bob to remove cards according to the following rules she has devised. During the process, she has her back turned and can not see what Bob is doing.

(1) Look at the value of the top card of the pack of cards in your hand hand. Subtract this value from 11 to give a number x.
(2) If there *are* enough cards in the pack, place the top card face down on the table (next to any cards already there) and discard the *next* x cards from the pack into a waste bin. If there are not enough cards to do this, go to step (4).
(3) If there are at least two cards in your hand, return to step 1.
(4) Ask Alice to turn round, and hand the remaining cards in your hand (including the top card) to her.

Alice now counts the face-down cards on the table (without touching them) and the cards in her hand, then tells Bob the total value of the cards that Bob has left face down on the table. How does she work this out?

ANSWER $12 \times$ (number of cards face down on table) + number of cards returned to Alice $- 40$

SOLUTION

At the end of the process, let:

$$\begin{aligned} b &= \text{number of cards in the bin} \\ d &= \text{number of cards face down on the table} \\ u &= \text{number cards returned to Alice} \\ T &= \text{total of cards face down on the table} \end{aligned}$$

If the value of the first card that Bob sees is n_1, then he must discard $11 - n_1$ cards into the bin. Similarly if the value of the second card that Bob sees is n_2, then he must discard another $11 - n_2$ cards into the bin. At the end of the process $n_1 + n_2 + \cdots = T$ so Bob has then discarded $11d - T$ cards.

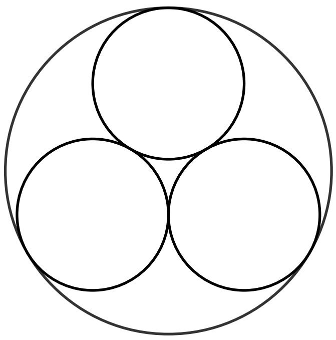
Given there are u cards remaining unused, we can also see that $b = 40 - d - u$.

Hence:

$$\begin{aligned} 40 - d - u &= 11d - T \\ T &= 12d + u - 40 \end{aligned}$$

> This reader will notice that this problem is only dependent on the total number of cards and the number chosen in rule (1).

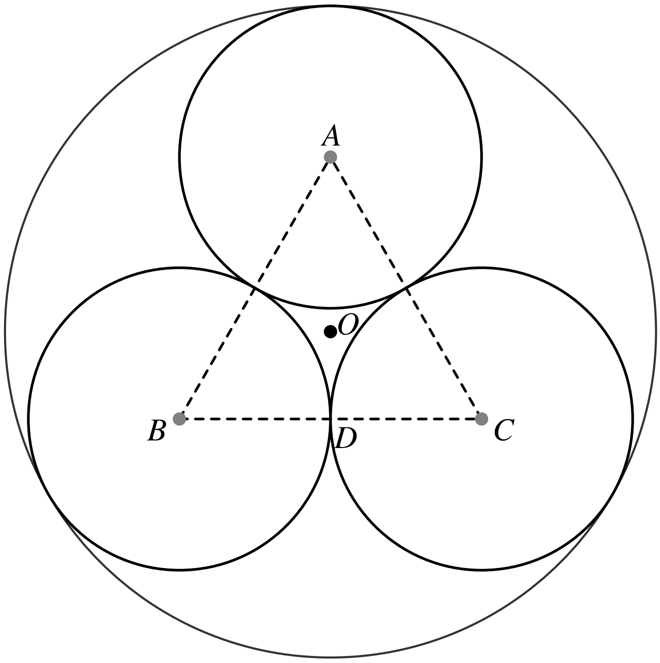
6. In the figure of four circles, the inner three circles with centres A, B and C have radius 1 unit. Find the radius of the outer circle incorporating $\sqrt{3}$ in your answer.



ANSWER $\frac{2}{3}\sqrt{3} + 1$

SOLUTION

First label the centres of the circles A, B, C and O as shown in the diagram. D is the midpoint of BC.



We are given that $AB = 2$ so using Pythagoras's theorem we can write

$$AB^2 = BD^2 + AD^2$$
$$2^2 = 1^2 + AD^2$$

so we find $AD = \sqrt{3}$. From the result of question 6(a) we now have $OA = \frac{2}{3}\sqrt{3}$.

Hence radius of the large circle = $\frac{2}{3}\sqrt{3} + 1$.

> There are many forms of the answer such as
>
> $$\frac{2}{\sqrt{3}} + 1, \quad \frac{2\sqrt{3} + 3}{3}, \quad \frac{2 + \sqrt{3}}{\sqrt{3}}$$

7. Show that there are values of $2^p - 1$ that are prime if p is a prime. What is the smallest prime p such that $2^p - 1$ is not prime? Also show that $2^{2p} - 1$ and $2^{4p} - 1$ can never be prime.

ANSWER 11

SOLUTION

For example, we see that $2^3 - 1 = 7$ which *is* prime.

$2^{11} - 1 = 2047 = 23 \times 89$.

Numbers like $2^p - 1$ which are prime when p is prime are called Mersenne primes. The second result shows that if p is prime, it does not guarantee that $2^p - 1$ is prime.

You will probably have looked up that $a^2 - b^2 = (a - b)(a + b)$. We can now choose what values or expressions a and b should take.

In the case of $2^{2p} - 1$, we should take $a = 2^p$ and $b = 1$. Hence

$$2^{2p} - 1 = (2^p - 1)(2^p + 1)$$

and both factors are greater than 1, so $2^{2p} - 1$ factorises and is not prime. Take $a = 2^{2p}$ to prove the second expression is not prime.

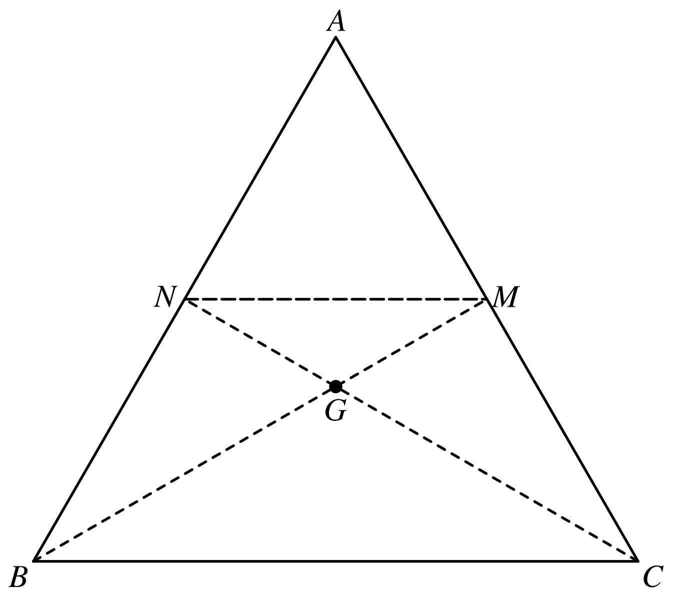
> This leads on to being able to prove that if p is not prime, then nor is $2^p - 1$ prime.

8. (a) ABC is an equilateral triangle. M is the midpoint of AC and N is the midpoint of AB. BM meets CN at G. Show that $BC = 2 \times MN$ and that $BG = 2 \times GM$.

(b) $ABCD$ is a regular tetrahedron. A plane passes through A parallel to $\triangle BCD$ (which is opposite A). There are three more planes through B, C and D defined in a similar way. These four planes intersect in pairs to give the edges of a larger tetrahedron; each vertex is the intersection of three of these planes. What is the ratio of the volume of $ABCD$ to the volume of the new tetrahedron?

Answer (b) 1 : 27

Solution



(a) Since M and N are respectively the midpoints of AC and AB, it follows that $\triangle ANM$ is similar to $\triangle ABC$ because the included $\angle BAC$ is common to both triangles. Hence $BC = 2 \times NM$ and NM is parallel to BC.

Because corresponding angles in $\triangle BCG$ and $\triangle MNG$ are equal (opposite angles at G and alternate angles between parallel lines), these triangles are similar. Hence $BG = 2 \times GM$.

We can say that G is a *point of trisection* of the line BM.

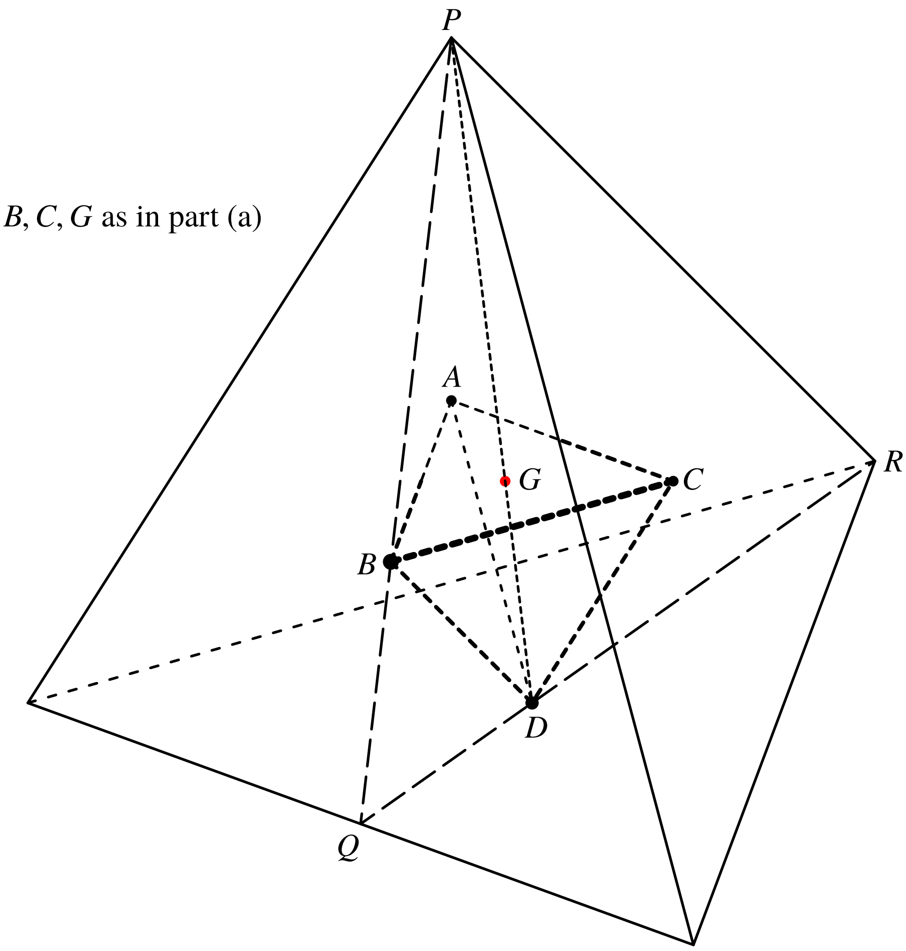
(b) It is easiest to work backwards when drawing this. Start with the larger tetrahedron then draw $ABCD$ inside it. By symmetry A, B, C and D must lie at the 'middle' of the triangular faces of

the larger tetrahedron, that is, the *equivalent* of G in part (a) for each outer triangle. To convince yourself of the working that follows, draw out the cross-section of one of the reflection planes of the tetrahedrons as shown, namely $PBQDR$.

From the result in (a) we see that $QB = \frac{1}{3}QP$ and $QD = \frac{1}{3}QR$. Hence $\triangle QBD$ is similar to $\triangle QPR$ and $\frac{1}{3}$ the size. Hence $PR = 3 \times BD$ and the tetrahedron has $3^3 = 27$ times the volume.

You may have to convince yourself of this last reasoning by considering the volumes of two cubes, one with edge 1, the other with edge 3.

labellings for A, B, C, G as in part (a)



cross-section $PBQDR$: